The Vanishing Remainders Paradoxes

And Other Overlooked Paradoxes of Infinity[1]

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Abstract

The rational 1/3 is transformed into the real (infinite decimal expansion) 0.333... by successively multiplying 1 by 10, dividing that 10 by 3 getting a quotient of 3 with a remainder of 1 which becomes the 10 for the next decimal place, and so on... But that remainder of 1 vanishes from theoretical view when we reach the Cantorianly completed countable infinity of decimal places. What happens to it? Does it somehow become an absolute zero after an infinite number of divisions? A variant asks: what happens to the remainder of 1 when it is divided by a natural number *n* as *n* "goes to infinity"? Does it somehow become an absolute zero when "divided by infinity"? These are some of the paradigmatic questions for the "Vanishing Remainders Paradoxes". If the remainder does not become an absolute zero, then 1/3 cannot be represented by 0.333..., and if it does, this violates an as yet unrecognized implicit conservation-type law, opening the door to inconsistency in addition to paradox. E.g., if the remainder of 1 when 1 is "divided by infinity" becomes an absolute zero, one can derive that the reals are countable. There are many other as yet overlooked paradoxes in mathematics, some of which offer a basis for their joint resolution, and for the problem of renormalization in physics.

Introduction

Infinity has been the paradigm of paradox since ancient times. Even the set theory developed by Cantor (Georg Ferdinand Ludwig Philip, 1845-1918) to formalize infinity, to give it mathematical rigor, to codify it and give it the status of law (for Cantor not just mathematical but religious), has been paradoxical since its inception. Joseph Warren Dauben (who encouraged this author with regard to The Good Shepherd's Paradoxes – see page 12 – for which many thanks), in his intellectual biography of Cantor (*Georg Cantor, His Mathematics and Philosophy of the Infinite*, 1979, Princeton University Press) gives us a fascinating look at the intellectual output of Georg Cantor, a mystic who wanted to find – or create – God, the ultimate paradox, in his mathematics of infinities. He lets us see in great detail that Cantor sought, not to describe God (even for Cantor this would be impossible), but to approach – and attain! – the absolute infinity of God through his theory of sets of successively greater "transfinities", perforce inherently "paradoxical".

[1] **MSC 2000 Classifications:** 03E35; 03E17, 03E25, 03E70, 03A05, 00A30

03-xx	Mathematical and Logical Foundations	03E25	Axiom of choice and related propositions
03Exx	Set Theory	03E70	Non-classical and second order set theories
03E35	Consistency and independence results	03A05	Philosophical and critical
03E17	Cardinal characteristics of the continuum	00A30	Philosophy of mathematics

In fact, with the acceptance of Cantor's set theory, paradox, which had earlier meant inconsistency in a mathematically and logically unacceptable sense, has come to be considered not merely acceptable, but conventional. With Cantor's work, infinity and its paradoxes left the realm of the philosopher and entered the world of mathematicians. The acceptance by mathematicians of this incongruous marriage of paradox and consistency in the two thirds of modern mathematics that has Cantor's set theory – along with logic – as an essential part of its foundations can be summed up in the rhetorical question posed by David Hilbert (1862-1943):

- ❋ "What mathematician would want to be expelled from the paradise which Cantor created?" (Hilbert)

So far...

One of the first of the new paradoxes was that Cantor showed that there was an "uncountable infinity" greater than the already accepted infinity of the natural or counting numbers (by definition a "countable infinity") by defining real numbers as (implicitly countably) infinite decimal expansions, then assuming one could make a list (again, countably infinite) of *all* real numbers, and then applying his famous and paradigmatic diagonalization construction to generate a new (and obviously valid) real number that could not be in this list. This gives a standard proof by contradiction that there are more real numbers than there are counting numbers; i.e. standard set theory holds that the set of real numbers is "uncountably infinite" or "non-denumerable".

(Note: the term "standard" is rarely needed or used in a mathematical work since everything is implicitly assumed to be "standard", even if new. But since this paper concerns "Overlooked Paradoxes", the term will be used frequently, to help remind or inform the reader of "which is which", that which is accepted as mathematically "standard" and should be familiar as such to a mathematician, and that which is likely to be unfamiliar since it has been "Overlooked".)

Many paradoxes, both ancient and modern, have been explored by mathematicians and philosophers over the millennia. Paradoxes of infinity have been especially popular. But... there are still many overlooked paradoxes of infinity in mathematics, especially in set theory and the real number theory based on it. The reader should find these paradoxes enjoyable to explore and analyze, but will definitely find them a challenge to resolve satisfactorily.

"The Vanishing Remainders Paradoxes" of real number theory are particularly accessible to all, even non-mathematicians, since they can be presented informally but compellingly. We will look at these and related paradoxes before moving on to the truly fundamental paradoxes, whose resolution may well inspire new theoretical foundations.

Fundamental but informally presented, "The Good Shepherd's Paradoxes" (page 12) are exceptionally easily accessible and extremely compelling, while their formal counterparts, "The Bijection Permutation Paradoxes" (pages 33-40), are especially cogent in the simplicity of their formal rigor. The more formal analysis of the latter (page 25) will offer a possibility of resolution for most, if not all, of these paradoxes.

The Vanishing Remainders Paradoxes of Real Number Theory

Straddling set theory and real number theory, a set of interrelated paradoxes come up when we closely reexamine the well-known and somewhat paradoxical result that all real numbers (usually based implicitly on the popular ZFC variant of Cantorian set theory) have a unique infinite decimal representation, *except* the integers. E.g. 4.999... = 5.000... are ostensibly 2 distinct but numerically equivalent infinite decimal expansion representations of the integer 5.

The key to this reexamination is to study how the infinite decimal representation of the rational number 1/3 is constructed.

Also key is some number theory:

❈ Assume that for integers a (≥ 0), b (> 0), q and r ($b > r \geq 0$) we have $a = b \cdot q + r$, where q is the quotient and r is the remainder that we get when we integer divide the dividend a by the divisor b. There is a number theoretical result that when we integer divide such a (non-negative) integer a by a positive integer b, the end result will be identically equal to (absolute) zero if and only if the quotient and remainder are both zero, or, equivalently, if and only if a was initially (absolute) zero. We also need to remember that $a / b = q + r / b$ when dealing with the integer values as real numbers, to maintain equality. If a strictly non-zero remainder r "Vanishes", then the strictly non-zero real number value r / b will have "Vanished", as well...

The "zeroth" decimal place (just to the left of the decimal point) of 1/3 is "0", i.e. the integer quotient of 1/3, the remainder being 1. (The "real number value" of the remainder at this point is "1/3", as should be obvious.) The first decimal place is obtained by multiplying this remainder of 1 by 10 (decimal) and again dividing by 3, giving us so far "0.3", the "3" being the quotient of 10/3, and the "continuing remainder" of 1. (The "real number value" of the remainder at this point is "1/(3 × 10)", again as should be obvious.) The succeeding decimal places are obtained in the same way, always multiplying the continuing remainder of 1 by 10 and dividing by 3, giving us "0.333...", for an infinity of decimal places. (It is important to remember that this infinity is a "completed infinity", Cantor's concept and fundamental in all modern set theories, as opposed to the merely "potential infinity" of the ancient, pre-Cantorian philosophers.)

But certain questions were never asked, such as:

Paradigmatic Questions for "The Vanishing Remainders Paradoxes"

1) In constructing e.g. 1/3 = 0.333... or 1/9 = 0.111..., what happens to that strictly non-zero, continuing remainder of 1?!
2) If it becomes a strict zero, completely "Vanishing", as real number theory seems to assume by ignoring the issue, just *how* does it do so?!

The questions sound satirical, which they are, but they also have very serious import:

3) If we divide 1 by n "enough times", does the continuing remainder of 1 become 0?!
4) If we divide 1 by n, does the remainder of 1 become 0 if n is "big enough"?!
5) Do the remainders "Vanish" just like the 1 in $\aleph_0 + 1 = \aleph_0$ does on the other side of the equation?

And we should also ask:

6) What happens to the remainder of 2 when constructing 2/3 = 0.666...?
7) What happens to the repeating sequence of non-zero remainders (3, 2, 6, 4, 5, 1,...) when we construct the infinite decimal expansion of e.g. 1/7 = 0.142857142857...?

And it's essential to ask:

8) What is 1 part in an infinity? or even in an uncountable infinity?

And whether we are referring to the continuing remainder of 1 when we construct 0.333... or the remainder of 1 when we construct $1/n$ (for n "big enough"), we NOTE:

- **If the remainder does *not* become a strictly absolute zero, then there is more to a real number than heretofore suspected.**
- **If the remainder *does* somehow become a strictly absolute zero, there is more to real number theory than heretofore suspected.**

E.g. the real number "0.333...", if derived from the rational number "1/3", must have a remainder of $1 \neq 0$ in the "last decimal place", which by standard theory doesn't exist (with a "real number value" of "$\frac{1}{3} \times 10^{\aleph_0}$" which doesn't exist either, since it would be an "infinitesimal" and set theory stoutly denies the existence of such beasts), allowing us to overlook and/or ignore the problem easily, or at least with plausible deniability. (This $1 \neq 0$ in fact does link us to the $1 = 0$ question that is raised by the paradoxical $\aleph_0 + 1 = \aleph_0$, on which more, later.) In our situation, a "normalized" remainder could conceivably be either 1, 2, or the 0 usually implicitly assumed, since we have a divisor-modulus 3 situation. If the remainder of 1 does *not* vanish, then when we multiply "1/3 = 0.333..." by 3, instead of 1.000... = 0.999... we actually get:

> "3 × 0.333... + 3 × the remainder of 1 in the 'last' decimal place, which last (3 × 1, now an un-normalized remainder) when normalized by dividing by the divisor of 3, gives a quotient of 1 and a remainder of 0 in the 'last' decimal place; this last 1 then needs to be "carried", i.e. added to 3 (the divisor again, now the multiplier) × the quotient of 10 / 3 (= 3, i.e. the value in the 'last' decimal place of 0.333...) = 9; and when that 1 is added to that 9 we get 10, i.e. an absolute 0 in the 'last' decimal place (absolute since the remainder in that place is 0, as well) with the 1 carrying into the 'next to the last' decimal place, which also doesn't exist theoretically, and so on ripple-carrying all the way back to and just past the decimal point giving us 1.000..."

I.e. we get:

DIVISOR × QUOTIENT	NON-EXISTENT "LAST DECIMAL PLACE"	+	*NON*-VANISHING NORMALIZED DIVISOR × REMAINDER	
3 × 0.3333333333333333333...	3	+	3 ×	1
			NVNR-CARRY	
= 0.9999999999999999999...	9	+	1	0
= 1.0000000000000000000...	0		0	0

Diagram 1: Multiply 3 × (1/3 "=") 0.333... with *Non*-Vanishing Remainder of 1

We get 1.000..., as we should expect to, and almost more importantly, we get 0 as our *Non*-Vanishing Remainder, also as we should expect to.

So, both satisfyingly and disturbingly, taking into account the previously Vanishing Remainder gives us a unique representation of the integers. Disturbingly because it means set theory and real number theory are both in trouble. For example, if 0.999... is distinct from 1.000... it distinctly raises the spectre of infinitesimals, the existence of which is still vehemently denied in standard theory. (Infinitesimals are included in non-standard analysis, but forcibly, not derived naturally.) On the bright side, it appears that these same infinitesimals are more naturally derivable than previously thought, if we can just "patch up the holes in the boat", as it were. Many have sought this San Graal.

The Vanishing Remainders Paradox 1: 1/3 = 0.333... and Its Remainder of 1

When converting the rational number 1/3 or 1/9 to a decimal real number, 1/3 = 0.333... or 1/9 = 0.111..., the process requires a strictly non-zero, continuing remainder of 1 for each of the countable infinity of decimal places. That remainder Paradoxically Vanishes from theoretical view when

all of the infinite number of decimal places have been constructed, as if it becomes an absolute zero.
WHAT HAPPENS TO IT?

It's obvious that a conservation law-like substance is violated if any *non-zero* Remainder Vanishes, and it's not a good idea to theoretically Overlook even a zero Vanishing.

The next Paradox is rather obvious, even at this early point:

The Vanishing Infinitesimals Paradox 1: Infinitesimals Vanish With the Vanishing Remainders.
When such Remainders Vanish, the possibility of any natural theoretical development of infinitesimals, or "transfinitesimals", Vanishes with it.

The Archimedean property figures in this prominently, on which more, later.

For some it goes without question that rational numbers are also real numbers. They are real by implicit fiat. It has never been questioned that they might not all have infinite decimal expansions (or some other base, with binary, octal and hexadecimal being currently popular). The same can be said of irrational numbers. No one questions that there is a 1-to-1 correspondence between infinite base expansions and all rational or irrational numbers. So one will find real numbers to be defined in both ways, as infinite base expansions, and as the rational numbers plus the irrational numbers.

The Vanishing Remainder also gives us the even more paradoxical result that:

Real Numbers Paradox 1: The Reals Do Not Contain The Rationals.
The real numbers, ℝ, as defined by infinite decimal (or any single base) expansions do not contain all the rationals. E.g. we cannot successfully represent either 1/3 or 1/9 as a infinite decimal expansion real number because of the strictly non-zero Non-Vanishing Remainder, although we can represent it as a ternary/base 3 real. **(See Real Numbers Paradox 3, page 6.)**

Following the trail of the Vanishing Remainder Paradoxes gets yet more complicated. If we now have a strictly non-zero "last decimal (or other base) place remainder (/divisor)", we have to be ready for it to have *any* normalized value (or perhaps any value, with normalization to follow), a range that is a function of the divisor. But if we had started with 1/9 instead of 1/3, the possible normalized remainders for the "last decimal place" would obviously be 0 through 8 (/9, divisor), but we would still have "0.999..." as the representation of both $3 \times 0.333...$ and $9 \times 0.111...$. Just looking at "0.999..." doesn't tell us what the remainders should be, not even whether there should be a remainder in e.g. the 0 to $3 \times (3 - 1) = 6$ range (with un-normalized remainders of 3-6 necessitating a carry to get them in the 0-2 range) or the 0 to $9 \times (9 - 1) = 72$ range (with un-normalized remainders of 9-72 also necessitating such a carry). We must also maintain information about the divisor, and the compound expressions get arbitrarily complex... but if we studied all the possible variations of such expressions for Vanishing Remainders – and Divisors – these Paradoxes would require a Galois to unravel them.

The Extended Vanishing Remainders/Divisors Paradox
If a Vanishing Remainder somehow stays Vanished, we might hold out hope that there may be no problem. But if it is ever sought (like the Grail) and hopefully found, we will soon notice that the Divisor, e.g. the 3 in 1/3, also Vanished and needs to be found. We also notice that there is no reason why the Remainder should need to be integer or to derive nicely from a rational number, even though it would need a Divisor-like substance to make sense.

We will quickly look at a subtle variant of the Vanishing Remainder in the "Un-Real Square Root of 2 Paradox" and the more general "Un-Real Irrationals Paradox".

The ancient Greeks proved that the square root of 2 was not a rational number. Number theory tells us that if the square root of 2 were rational then we could write $2 = (a/b)^2$ and $2b^2 = a^2$. It also tells us that both sides of this second equation must have the same prime decomposition, and also must be perfect squares since the right side of the equation is a perfect square. Except for the first 2, both sides of the second equation are perfect squares, but the extra 2 on the left hand side of the equation means that side cannot be a perfect square unless the prime number 2 is itself a perfect square. So number theory tells us is that the square root of 2 cannot be a rational number, in the sense of a ratio of finite positive integers, unless the prime number 2 just happens to (also) have a non-trivial prime decomposition (and a square one at that). Since it doesn't, the square root of 2 cannot be a rational number.

But, if we look at an extended rational approximation of the square root of 2 in decimal representation we get: $2 = \left(\frac{14142...}{10000...}\right)^2$ which we can rewrite as $2 \times (10000...)^2 = (14142...)^2$. If we had an infinite number (a "completed infinity") of such decimal places, real number theory would hold that we had the *precise* square root of 2 and not merely an approximation. But overlooked was the paradoxical question:

Un-Real Square Root of 2 Paradox: A Non-Trivial Prime Decomposition of 2?

How many decimal places of the real decimal expansion of the square root of 2 are required to give the prime number 2 a non-trivial prime decomposition (and a square decomposition, at that)? In actual fact, not even Cantor's absolute infinity can give us enough decimal places to allow 2 to have a non-trivial prime decomposition and thus to allow the square root of 2 to be a standard decimal expansion real number. (This of course also applies to any base other than decimal, and probably to all other irrationals as well.)

This question/argument has the same compelling quality of the ancient Greek argument for the irrational nature of the square root of 2. We can optimistically generalize the Un-Real Square Root of 2 Paradox to get the:

Un-Real Irrationals Paradox: No Irrational Numbers Are Real Numbers.

E.g. when we try to turn the rational approximation of the square root of 2 into a real number (infinite decimal or other base expansion), we need enough decimal places to give the prime number 2 a non-trivial prime decomposition, in fact a square one. I.e. the square root of 2 and (probably, if we want to hedge) all other irrational numbers cannot be standardly defined real numbers, of any base, even all possible bases taken together. (Galois could prove this general case handily.)

and, for emphasis, the:

Real Numbers Paradox 2: The Reals Do Not Contain The Irrationals.

The real numbers as standardly defined by infinite decimal expansions do not contain the irrationals (not any, probably; not even using all bases combined, probably... hedging, to be safe).

Going back to Real Numbers Paradox 1, we saw that the rational 1/3 was contained in the ternary reals but not the decimal reals. I.e. if we look at the infinite ternary (base 3) representation of 1/3, we find that the Vanishing Remainder problem itself Vanishes, at least temporarily. The rational number 1/3 *can* be completely described by the infinite ternary expansion 0.1000... with an absolute 0 remainder that does not embarrass us if it "Vanishes". That is, 1/3 can be made a ternary real number with not even the "infinitesimal" loss of accuracy associated with a (non-zero) vanishing remainder. We now find that there are ***different** **sets** $\mathbb{R}$ **of** **real** **numbers***, as standardly defined by infinite base expansions, highly dependent on the bases used in a way obviously relating to the prime decompositions of the bases.

It is pretty easy to figure out that:

- ❋ If each base has at least one prime in its decomposition not shared by the other base, the sets of reals will be at least "partially incommensurate" with respect to the other, i.e. neither set will completely contain the other. (If we think of a set of reals that is contained in another set of reals as "commensurate" with that set, "commensuration" and "incommensuration" will in general be asymmetrical, i.e. not commutative; e.g. the set of binary reals (primes = {2}) is "commensurate" with the decimal reals (primes = {2,5}), since all binary reals can be completely represented as decimal reals without Vanishing Remainders, but the decimal reals are "incommensurate" with the binary reals, since some decimal reals cannot be represented as binary reals without Vanishing Remainders.) If the bases share no primes, their respective reals will be completely different sets except for their integers, and they will each be "completely incommensurate" with respect to the other.
- ❋ The reals base n will contain the reals base m (thus making the latter commensurate with the former) if and only if all the primes of the prime decomposition of m are contained in the set of such primes for n (but allowing arbitrary exponents/powers of primes).
- ❋ The reals base n and the reals base m will be the same set of numbers if and only if the "prime containment" (here ambiguous as to "in" or "by") and thus "commensuration" go both ways, i.e. the reals will be "completely commensurate" (e.g. bases 10, 50 and 100 reals all have primes = {2,5}).
- ❋ We need to remember that the standard reals are really just an extension of the rationals to transfinite numbers, i.e. rational numbers with "infinite numerators" and "infinite denominators".

So, we get yet another Real Numbers Paradox:

Real Numbers Paradox 3: The Reals Do Not Contain the Reals.

The real numbers, $\mathbb{R}$, as defined by infinite decimal (or any other single base) expansions do not even contain (all) the reals. E.g. the set of ternary reals is not contained in the set of decimal reals, and vice-versa. In general containment will depend on the prime decompositions of the bases.

We are also reminded, since $3^{\aleph_0}$ here (as opposed to in standard theory) seems to be necessarily different from $10^{\aleph_0}$, that it was never really that obvious why $2^{\aleph_0}$ was the cardinality of the "continuum", and not e.g. $2^{2^{\aleph_0}}$.

We will expand on all this somewhat by proposing the "Quantinuum Hypothesis" in a later section (see page 45).

We can also note for later that the standard proof that e.g. 4.999... = 5.000... involves multiplying 4.999... by 10 and then subtracting the original 4.999... (ostensibly) giving us 45.000... which we then divide by 9 giving us the 5.000... we saw above. The multiplying by 10 has an essential correspondence to an $n \leftrightarrow n + 1$ mapping, essential in set theory, since it maps the nth decimal place onto the $(n-1)^{st}$ decimal place, for all counting numbers n. This process, too, overlooks the "Vanishing Remainder", the importance of which will be seen later.

More Vanishing Remainders Paradoxes

(If the following starts seeming too technical, feel free to skip to "The Good Shepherd's Paradoxes", page 12, which address question 5 of the **Paradigmatic Questions for "The Vanishing Remainders Paradoxes"**, page 3, concerning the Vanishing 1 of $\aleph_0 + 1 = \aleph_0$. These Paradoxes reveal the source of the Vanishing Remainders Paradoxes, and are far more fundamental to a viable systemic resolution.)

Remainders Vanish in circumstances other than constructing real numbers from standard rational numbers. Earlier (page 3) we asked the question; "If we divide 1 by *n*, does the remainder of 1 become 0 if *n* is 'big enough'?!" This is in fact another "first order" Vanishing Remainders Paradox:

The Vanishing Remainders Paradox 2: The $\lim_{n\to\infty} 1/n = 0$ and Its Remainder of 1

When taking the standard limit $\lim_{n\to\infty} 1/n = 0$, the remainder of 1 also Vanishes.

WHAT HAPPENS TO IT?

The Vanishing Remainders Paradoxes actually form an extended family of interrelated families of paradoxes, which we have only just started to explore. Others include the "Countable Reals Paradoxes", which along with their counterpoint, the "Non-Denumerable Rationals Paradox" (see page 42), also straddle set theory and real number theory. Also important is "Continuum Hypothesis Paradox 1" (page 10) that we get from deriving countable reals using a denumerable partition of the unit interval.

If we look at the "limit" of the closed interval [0,1/*n*] "as $n \to \infty$ ('approaches' infinity)", by standard theory we get $\lim_{n\to\infty}[0,1/n] = [0,\lim_{n\to\infty} 1/n] = [0]$, an interval/set with only one number/point in it, a set of "measure zero". Standard real number theory and standard measure theory have both ignored the "Vanishing Remainder", in this case when *n* is "big enough".

When we ignore this Vanishing Remainder we get the further Paradox that, if we look at the countable equi-interval closed cover (not a partition since the intervals are not disjoint) of the unit interval [0,1/*n*],[1/*n*,2/*n*],...,[(*n*-1)/*n*,1], we find that "as $n \to \infty$", i.e. "in the limit", we get a countably infinite (denumerable), equi-interval closed cover for the unit interval, each closed interval of which provably has at most 1 real number in it (though each should standardly have $\sim 2^{\aleph_0}$ numbers/points in it).

- **Since the infinities of set theory are Cantorianly "completed" as opposed to merely "potential", "in the limit" goes beyond merely saying that "we can only get closer and closer, but never all the way there" all the way to "the Eagle has landed."**

The paradox is that this means there can (provably) be at most countable real numbers in the unit interval (easily extendable to the entire real number line), in counterpoint to the standardly accepted non-denumerable (uncountably infinite) real numbers.

The Non-Archimedean Reals Paradox

The real numbers are standardly Archimedean, and the Archimedean property means that numbers cannot be infinitesimal. (Zero is also non-Archimedean.) It is Paradoxically Overlooked that it also means they cannot be infinite, either. Also, real number theory should strictly restrict itself to real numbers, e.g. in real arithmetic expressions. A common situation where it doesn't regards e.g. the "limit 1/*n* as $n \to \infty$". Since "infinity" must be a Cantorianly completed infinity, when taking such a limit, the real variable *n* takes on a "un-real" transfinite value. So the standard usage of taking limits

at infinity is actually strictly "extra-theoretical", technically, i.e. strictly "iffy-theoretical", even though we do it all the time. And thus real numbers, Paradoxically, are not as strictly Archimedean as one might think. In fact, having infinite numbers/points in a finite interval (e.g. the unit interval) means that at least some of the strictly non-zero distances among them can't be standardly Archimedean (e.g. around "cluster points").

This is the kind of thing (see also page 43) that, especially when combined with a Vanishing Remainder, can give rise to many... Overlooked Paradoxes, such as the:

Countable Reals Paradox 1

From the countable closed cover of the real number line, we find that the real numbers, ℝ, are *at most* countably infinite (as seen above; reminder: this is in addition to standard proofs that there are uncountably infinite reals).

and:

Countable Reals Paradox 2

From that same countable closed cover of the real number line, we also find that the cardinality of the set of real numbers, ℝ, is 1.

These are merely some of the initial aspects of the family of "Countable Reals Paradoxes" Overlooked so far by the mathematical community. The situation gets even more interesting, Paradoxically speaking.

Each closed interval in the cover shares precisely 1 point with each of its neighbors, and this gives us a real mess. Each closed interval, e.g. [1] or [0], has only 1 number to share. By induction we not only get that the unit interval can only have 1 real number in it, but we get at least 2 obvious candidates, 1 and 0 (and one of many Paradoxical, yet formalizable, proofs that 1 = 0). Further, it is trivial to extend this to the entire real number line, giving us at most 1 real number in .

The limit as $n \to \infty$ of $[0,1/n),[1/n,2/n),...,[(n\text{-}1)/n,1)$, a denumerable partition of the unit interval, is yet more paradoxical, especially when extended to the entire real number line. The partition intervals of real numbers are each associated with a unique natural number. (See Diagram 2, just below.)

The natural number 1 corresponds to the paradoxical interval [0); all the standard natural numbers, the "finite" ones, correspond to intervals "*extremely* close to [0)"; and $\aleph_0$ corresponds to the "last" interval, [1) (or [1] if we don't extend the partition). But the intervals "not close to" [0) but not precisely [1), either, what numbers do they associate with?! In between [0) and [1) are many doubly paradoxical intervals. They represent numbers distinctly greater than "finite", but also distinctly less than the standardly "first" transfinite number/cardinal, "$\aleph_0$". Informally, at this stage, we seem to be able to speak e.g. of "$\aleph_0/2$", "$3\aleph_0/2$" (out past 1 and therefore distinctly greater than $\aleph_0$, but also distinctly much much less than $2^{\aleph_0}$), etc. By standard, accepted theory none of them exist, yet Paradoxically they also *must* exist, Overlooked by standard theory.

1... ?! ?! ?! ?! ?! ?! $\aleph_0$... ?! ?! ?!

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[0) [0.250...) [0.500...) [0.750...) [1.000...) [1.250...) [1.500...)

Diagram 2: A Denumerable Equi-Interval Partition of the Real Number Line.
The intervals of a denumerable equi-interval partition of the real number line correspond to natural numbers or cardinals: [0) to 1, intervals "extremely close to [0)" to all standardly finite natural numbers, and [1) to $\aleph_0$; but what about the infinity of other intervals, e.g. [0.500...) or [1.500...)?!

In any case, we find the first of the Continuum Hypothesis Paradoxes:

Continuum Hypothesis Paradox 1: Transfinite Cardinalities Less Than $\aleph_0$?

There seem to be many distinct infinities of cardinality less than $\aleph_0$, and many distinct infinities of cardinality greater than $\aleph_0$ but less than $2^{\aleph_0}$, derived from a denumerable equi-interval partition of the unit interval... and from a Vanishing Remainder.

A new, perforce non-standardly non-Cantorian set theory would need to live with such entities non-paradoxically, perhaps by conceiving of "infinity" (of the "neo-natural" numbers) as "fuzzy", or "relativistic". NOTE that this gives us a possible paradigm for making transfinite numbers and "transfinitesimals" Archimedean. (More on this, page 43.)

We also have at least three other Paradoxes that jump out at us:

Semi-Open and Open Intervals Paradox

What is the $\lim_{n\to\infty}[0,1/n) = [0,\lim_{n\to\infty}1/n)$ or the $\lim_{n\to\infty}(0,1/n) = (0,\lim_{n\to\infty}1/n)$?!

How many numbers/points are there in the semi-open interval [0)?! or the open interval (0)?! There seem to be *none* since [0) is $\{\, r \mid 0 \le r < 0 \,\}$, i.e. the set of reals both greater than or equal to 0 and strictly less than 0, and (0) is $\{\, r \mid 0 < r < 0 \,\}$.

The Vanishing Infinitesimals Paradox 2: $\lim_{n\to\infty}[0,1/n) = [0,\lim_{n\to\infty}1/n) = [0)$?

The interval $\lim_{n\to\infty}[0,1/n) = [0,\lim_{n\to\infty}1/n) = [0)$ Paradoxically should contain $\sim 2^{\aleph_0}$ real numbers (with no rational numbers; see "More Real Number Theory Paradoxes", page 41) instead of... well, seemingly not *any* real numbers, i.e. $0 \le r < 0$: "all" real numbers that are *both* greater than or equal to 0 *and* strictly less than 0. We get a similar result for the limit of (0,1/*n*). This gives us the double Paradox that not only must there exist non-zero Vanishing Infinitesimals, contradicting standard theory that says they *cannot* exist, but there must also exist multiple orders of infinitesimals, or "transfinitesimals", corresponding somewhat to the various orders of transfinite cardinals.

Countable Reals Paradox 3

If the entire real number line is partitioned (as above) with equal intervals of length $\lim_{n\to\infty}[0,1/n) = [0)$, then there can be NO real numbers.

The "Countable Reals Paradoxes"...

Mild Digression: One question that comes up is "what infinity does *n* go to?!" (A similar question arises in automata theory, where there is also the – usually unasked – question of the halting of a Turing machine after it has written an "infinite" number of ones on its tape, so as to correspond to Cantor's idea of "completed infinity". Mathematicians have never been as careful as they should be when speaking of "infinity" and the "infinite" to distinguish among the "complete", the "completeable", and the "uncompleteable", the denumerable and the non-denumerable.) If, instead of the usually implied $\aleph_0$, our *n* goes to $2^{\aleph_0}$, we can fend off some of the paradoxes temporarily. But in the long run they will still catch up with us. End of digression.

There are further serious repercussions that derive from the Vanishing Remainders Paradoxes, but they are much more technical and will be postponed till after a highly accessible look at The Good Shepherd's Paradoxes and their formal counterparts, The Bijection Permutation Paradoxes. (See "More Real Number Theory Paradoxes" on page 41 and, for a look at a future alternative to the Continuum Hypothesis, "The Quantinuum Hypothesis" on page 45.)

"The Good Shepherd's Paradoxes"

Now we get to some new paradoxes in set theory proper, subtly related to the Vanishing Remainder Paradoxes, but more fundamental, and more revealing of the "source of the Nile". We still examine a Vanishing Act, but this time the 1 that Vanishes is the 1 in that acceptedly paradoxical and paradigmatically fundamental equation of standard transfinite arithmetic, $\aleph_0 + 1 = \aleph_0$.

Mathematicians generally mistrust physical analogies as proofs in mathematics (probably for reasons relating to Löwenheim-Skolem), but here we will use physical analogies because they make the paradoxes so much clearer and accessible to all. Once the essential insights are found to be "intuitively obvious", the way to make them amply formally rigorous will be equally intuitively obvious.

Historically, this next paradox has been referred to as "**The Golfer's Paradox**" (page 13), which along with "**The Banker's Paradox**" (page 16) constitute two of "**The Good Shepherd's Paradoxes**". We will forgo the historical details to concentrate on the relevant fundamentals.

Imagine that we have a small ball or coin that is "homeless", and in our case here, a small ball paired with a small box that can hold precisely one such ball, as seen in Figure 1. (The Golfer, a Scotsman, originally used golf balls and "wee small glasses" for his demonstration. He was offering a challenge to a Mathematician, a Banker, and a Man of the Cloth during a Game of Golf, at the 19th Hole.)

$(\oplus)$ $\|(\otimes)\|$

Figure 1: 1 homeless ball 1 ball paired 1-to-1 with/in 1 small box

If for some reason we wanted to put *both* the homeless ball on the left *together with the ball that is in the box* into a precisely 1-to-1 correspondence *with* the box, it is "intuitively obvious" that we cannot do so. In fact, it is difficult to imagine anyone who would think that it could be done (the square root of -1 to the contrary notwithstanding).

A formal analysis would note that merely switching the balls would yield a situation that was abstractly equivalent to the pre-switching situation with regard to our concern here: constructing the desired 1-to-1 correspondence. I.e. it wouldn't help even a little: no matter how many times the balls were switched, they could never be put into a 1-to-1 correspondence with the 1 box, not even if they were switched a transfinite number of times, not even if they were switched a Cantorianly absolute infinity (the largest standard set theory allows) of times.

If we try to make the homeless ball, or any other ball, "Vanish", we are Cheating. (If we think of it as a coin, we can call this "Embezzling".) For example, if for any reason the 2 balls "become 1", as standard paradoxical measure theory seems to allow, that would be Cheating. Or if we supply an additional empty box, we are Cheating. And worse yet, we do not get the desired result even if we do Cheat. (Such is always the case with actual cheating: once you cheat, you can never again win The Game.)

Now let us consider whether we can improve the situation (in Figure 1) by adding another ball-in-a-box pair. When we do, we now have a homeless ball and 2 small boxes, again paired or re-paired with precisely 1 ball each.

(⊕) ‖(⊗)‖ ‖(⊗)‖

Figure 2: 1 homeless ball 2 balls paired 1-to-1 with/in 2 small boxes

Now if for some reason we wanted to put the homeless ball on the left *together with the 2 balls that are in the 2 boxes* into a precisely 1-to-1 correspondence *with* the 2 boxes, it is again intuitively obvious that we cannot do so. A formal analysis would include notice of the fact that the second ball-box pair is abstractly equivalent (for our purposes here) to the first pair, so you might as well switch with the first paired golf ball, and switching any number of times is again of no avail, and no matter what order we might try to switch them in (2^{nd} ball-box, 1^{st} ball-box,...), and adding it can't help. The abstract equivalence makes it immediately apparent that adding "successive" ball-box pairs will never be of any avail. It is essential to NOTE:

➢ **"The Handwriting is on the Wall."**

We can also note that, at this point in our analysis, it is of no import that we have not numbered either the balls or the boxes. (This will change for the next paradox.)

(⊕) ‖(⊗)‖ ‖(⊗)‖ ‖(⊗)‖ ‖(⊗)‖ ‖(⊗)‖ ...

Figure 3: 1 homeless ball infinity of balls paired 1-to-1 with/in boxes...

Now we must ask:

The Golfer's Paradox and its Paradigmatic Question

o **How many ball-and-box pairs do we need so that we can put the homeless ball, along with *all* the other balls that are paired precisely 1-to-1 with/in the boxes, into a precisely 1-to-1 correspondence with *all* the boxes?! Without Cheating...**

To reiterate, a careful formal analysis will have noted that, with regard to our purposes here:

❋ **each and every homeless ball (we may have more than 1) is abstractly equivalent to every other homeless ball, e.g. any newly homeless ball is abstractly equivalent any previously homeless ball (at the time of its homelessness);**

❋ **each and every paired ball is abstractly equivalent to every other paired ball;**

❋ **each and every box (all are paired) is abstractly equivalent to every other box;**

❋ **each and every ball-box pair is abstractly equivalent to every other pair, so e.g. one might as well switch the homeless ball with the first paired ball;**

❋ **adding a ball-box pair is abstractly equivalent to not adding it;**

❋ **in fact, adding any number of "successive" ball-box pairs is abstractly equivalent to not adding them;**

❋ **so, just 1 ball-box pair (or even none at all) is abstractly equivalent to any number of pairs;**

❋ **switching the currently homeless ball with any paired ball is abstractly equivalent to not switching them;**

- **in fact, switching the currently homeless ball with any paired ball *any number of times* is abstractly equivalent to not switching them;**
- **so, just one switch (or even none at all) is abstractly equivalent to any number of switches;**
- **each and every box is already paired with its maximum (and minimum) number of balls, 1;**
- **there are no empty boxes and adding an empty box is Cheating;**
- **the homeless ball refuses to vanish and "Vanishing" it is also Cheating;**
- **and, "set theory's only hope", "simultaneously" lifting *all* the infinity of balls out of *all* the infinity of boxes, shoving them over one place to the right leaving the first box empty, and then reinserting them in the remaining infinity - 1 boxes, is... iffy-theoretical in the extreme. (And The Banker's Paradox will falsify this approach.)**

(This is all made more obvious by our not numbering ball-box pairs at this stage. Also, see "A Review Of the Use of Induction, Finite and Transfinite", page 22.)

- **The reason The Golfer's Paradox can properly be called a Paradox is that set theory says that we *<u>can</u>* do it: set theory says that we *<u>can</u>* pair the infinity + 1 of golf balls with the infinity of boxes, in obvious contrast to the Paradoxical demonstration by our Scottish Golfer that it can *not* be done.**

I.e. standard set theory says that if we have an infinity of balls paired 1-to-1 with/in boxes, then we can put the homeless ball along with the rest of the infinity of balls into a 1-to-1 correspondence with the infinity of boxes. Cantor used a process, accepted to this day, analogous to switching the homeless ball with the ball in the first box, switching the new homeless ball with the ball in the second box, etc. until we have them "all paired". Since there is always another box in the infinite sequence of boxes with balls, he reasoned, there is a 1-to-1 correspondence between the balls and the boxes that does not exist if there are only a finite number of boxes with balls.

But this process cannot be "completed", even though standard set theory maintains that it *can* (mostly by leaving the assumption implicit and unquestioned). Balls (set elements) in fact must be "juggled", not just an "infinite" number of times and then "stop" with all the balls paired 1-to-1 with the boxes, but a truly un-completeable "forever". And this is Cheating.

(This is in fact what happens to the guests staying at the famous Hilbert Hotel, imagined by David Hilbert. They timeshare, each – at first – spending an "infinitesimal" fraction of time being juggled between rooms. As more infinities of guests are added, each guest spends a larger fraction of the time "up in the air" until there are so many that each only spends an "infinitesimal" fraction of the time actually accommodated in a room. See also "The Bijection Permutation Paradox 1, 'Counting' and Cantorian 'Reordering", page 36.)

Perhaps Cantor et al Overlooked that this process must become a *completed* infinity of switches, but cannot be since set elements must be juggled forever *in*-completely... perhaps... Who knows *what* they were thinking? In any case, the physical analogy of The Golfer's Paradox (page 13) clearly demonstrates this fallacy in Cantor's and set theory's formal reasoning.

We should also note that it comes up in arguments that have been offered by professional mathematicians that the 1-to-1 correspondence that Cantor had in mind has a "simultaneous existence" that no longer depends on the initial serial construction of the natural numbers, and therefore we should be able to "simultaneously" lift all the infinity of balls out of all the infinity of boxes, shove them over 1 place to the right, and reinsert them in the boxes. (We should NOTE that Cantor made no mention of any need for the re-mappings of the 1-to-1 correspondences to be "simultaneous".) This "simultaneous existence" is – how shall we put it? – "extra-theoretical". It has not, and cannot since they give such different results, be formally proven that a "simultaneous existence" can be consistent, as it is required to be, with the initial serial construction of the natural numbers and all the theorems that can be derived there from, per standard theory. Some people also think of the reals as having such a "simultaneous existence" with a corresponding "simultaneous construction", probably because it is hard to imagine how to construct them all serially in terms of current theory.

Now we get to The Banker's Paradox, "the other side of the coin" of The Golfer's Paradox. It is especially important because it can more readily be made compellingly formal than The Golfer's Paradox. (See pages 34 and 39 for The Bijection Permutation Paradoxes, the formal, set theoretical variants of The Golfer's and Banker's Paradoxes.)

In The Banker's Paradox it becomes necessary to start using numbering, but the Banker prefers to refer to the balls and boxes as "coins and slots". All coins, and all slots, are uniquely numbered, each pair initially numbered the same. (Figure 4.)

$$(0) \qquad \left\|\frac{(1)}{1}\right\| \left\|\frac{(2)}{2}\right\| \left\|\frac{(3)}{3}\right\| \left\|\frac{(4)}{4}\right\| \left\|\frac{(5)}{5}\right\| \cdots$$

Figure 4: homeless coin 0 infinity of paired numbered coins in slots...

Now we assume that homeless coin 0, along with all the other uniquely numbered coins (1,2,3...), are somehow put into some arbitrary 1-to-1 correspondence with the uniquely numbered slots (again, 1,2,3..., with no slot numbered 0). (Figure 5.)

$$\left\|\frac{(j)}{1}\right\| \cdots \left\|\frac{(m)}{n}\right\| \cdots \left\|\frac{(p)}{q}\right\| \cdots \left\|\frac{(1)}{k}\right\| \cdots \left\|\frac{(s)}{r}\right\| \cdots$$

Figure 5: infinity of paired numbered coins in slots, all mixed up...

While maintaining invariant a global 1-to-1 correspondence (set theoretically, the "bijectivity" of the "bijection"), we can switch the coin numbered 1 that is in slot *k* with the coin numbered *j* in slot 1, so that we have coin 1 in slot 1 and coin *j* in slot *k*. (See Figure 6.) This switch only involves 2 pairs of coins and slots, so it is trivial to verify that it preserves 1-to-1-ness ("bijectivity") globally, i.e. over the whole set of paired coins and slots. It is also trivial to verify that it can not possibly deprive a coin of a slot.

$$\left\|\frac{(1)}{1}\right\| \cdots \left\|\frac{(m)}{n}\right\| \cdots \left\|\frac{(p)}{q}\right\| \cdots \left\|\frac{(j)}{k}\right\| \cdots \left\|\frac{(s)}{r}\right\| \cdots$$

Figure 6: infinity of coin-slot pairs, all mixed up except coin 1-slot 1...

It is easy to see that we can continue this for 2,3,.... (We only need to apply finite induction, or transfinite induction, if necessary.) In fact, all coins that have a slot with the same number (i.e. 1,2,3...) can be re-paired in this manner with that slot without adversely affecting the global 1-to-1 correspondence, without depriving a coin of a slot. (It is essential to note that this procedure will neither make an initially valid 1-to-1 correspondence invalid, nor make an initially "invalid 1-to-1 correspondence" valid. The reason this is essential is immediately obvious.) The Banker calls this re-pairing procedure "Auditing", since it ensures that all the initially paired coins and slots are present and accounted for.

But then we get the disturbing question raised by Figure 7.

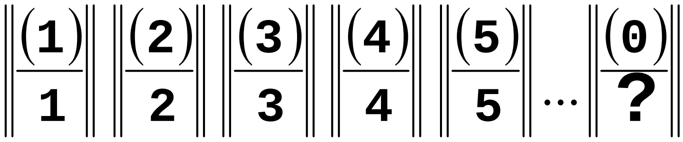

Figure 7: infinity of numbered coin-slot pairs, all re-paired... except 0?...

We must now ask:

The Banker's Paradox and its Paradigmatic Question

- **If all coins numbered 1,2,3... have been re-paired with their corresponding slots 1,2,3... (i.e. they *all* again share the same uniquely corresponding numbers) using a procedure that cannot possibly deprive a coin of a slot, what slot is coin 0 in?!**

(Amazing as it may seem, at this point many professional mathematicians have said things like: "finite induction only proves that you can re-pair a finite number of coins", or "the assumption that you can re-pair all the coins is proven false because you get a contradiction", or "the sequence (of the permutations of the bijections) doesn't converge" (as if that convergence wasn't theoretically required); some *professional mathematicians*, faced with this paradox, have even said "you can't make an infinite number of coin switches because it would take forever." These are some of the historical reasons for naming these "The Good Shepherd's Paradoxes".)

No such switching of coins in slots can deprive a coin of a slot or adversely affect the global 1-to-1 correspondence (except of course in the particulars of which coin is in which slot, which particulars, for our purposes here, have been *harmlessly* abstracted out). If in fact there was an initial precisely 1-to-1 correspondence between them, then there will be a precisely 1-to-1 correspondence after *all* such switching. But it is intuitively obvious that coin 0 cannot be in slot 1, nor in slot 2, nor slot 3..., so what slot can it be in?! The extreme generality and arbitrariness of the (ostensible) 1-to-1 correspondence in the above approach help make it apparent that we can *never* in fact have a 1-to-1 correspondence between coins 0,1,2,3... and slots 1,2,3... (and that this applies generally to set theory; this will be made formal with the Bijection Permutation Paradoxes, starting on page 25).

Embezzling coin 0 ("Vanishing" it, or any other) is Cheating. Providing a new empty slot is also Cheating. Besides, what number could that new empty slot have?! If it has 0, we will Get Caught; if it has 1,2,3... we will Get Caught; if... well, the picture is clear.

Cheating here has serious consequences (speaking now with set theory terms mixed in): if we Embezzle a coin from {0,1,2,3...}, then we have made that set "non-self-equal", i.e. not equal to itself; if we provide a new empty slot, we have done the same with the set {1,2,3...}. Even if we provide a new paired coin and slot (which doesn't help, but is also Cheating), we introduce non-self-equality and non-self-equivalence of sets, just as we would if we added a new number to the set $\mathbb{N} \equiv \{1,2,3...\}$ of all natural numbers. (In fact, recently, some mathematicians have tried to redefine the set of all natural numbers as $\mathbb{N} \equiv \{0,1,2,3...\}$. This makes sense since we should be able to count zero entities, decimal notation and all that. But it is humorous in light of our situation here.)

It is intuitively obvious that there could not have been a valid initial 1-to-1 correspondence between coins 0,1,2,3... and slots 1,2,3..., not even if the "simultaneous" lifting-shoving-reinserting method has been used. It is also easy to see how The Golfer's Paradox and The Banker's Paradox are twins, complements of each other. Historically they were jointly given the name of "The Good Shepherd's Paradox" since they establish that even a mere 1 added "sheep" can never truly be a "lost sheep", even in an absolute infinity of sheep. It is also clear how this all applies to set theory. Set theory's $\aleph_0 + 1 = \aleph_0$ guarantees that there can be literally *any* number of "lost sheep", (for a theoretically un-completeable) "forever", which explains why the problem of "renormalization" in physics is so intractable, despite physicists having found temporarily workable "solutions".

Since The Good Shepherd's Paradoxes have been expressed in physical analogs, even though they are intuitively compelling ones, and since the combined result is so... far reaching, some mathematicians will feel an intense need for formal rigor. Because of the seriousness of the consequences, they may not trust that overwhelming sense of IOTETMCO (Intuitively Obvious To Even The Most Casual Observer) that pervades this situation. Hopefully sufficient formality will be provided in the following sections, where The Good Shepherd's Paradoxes will be formally and set theoretically reframed as "The Bijection Permutation Paradoxes".

Introduction to a More Formal Analysis of the Primary Paradoxes

The reader who has carefully analyzed and appreciated The Vanishing Remainders Paradoxes, which are at the very least disquieting, and The Good Shepherd's Paradoxes in particular, which are outright distressing, will already be aware that set theory is "in trouble", and will not find the rest of this paper as presumptuous or off the wall as it might otherwise at first appear.

The Vanishing Remainders Paradoxes and The Good Shepherd's Paradoxes are extremely compelling intuitively, but the extreme seriousness of the consequences for set theory suggests that these paradoxes be expressed and analyzed with more formal rigor. Further analysis of the Vanishing Remainders Paradoxes really involves detailing "difficulties" in real number theory, such as demonstrating that infinitesimals or "transfinitesimals" are inevitable if transfinites are allowed, as well as other problems with real numbers. This should be more than sufficient to make clear that real number theory has serious failings in its set theory foundations. (See page 41 for "More Real Number Theory Paradoxes".) We have also glimpsed the source of the Banach-Tarski Paradox of (standard) paradoxical measure theory.

Infinitesimals have long had an alluring fascination for mathematicians, and surely pursuing them would be an enjoyable pursuit. But The Good Shepherd's Paradoxes get at the roots of the foundations of set theory in a way that the Vanishing Remainders Paradoxes do not, so the formal analyses that follow will favor them almost exclusively. Before proceeding, more introductory remarks are in order.

- **"*All* mathematicians believe it is *theoretically* possible that set theory is inconsistent. *No* mathematicians believe it is *actually* possible that set theory is inconsistent."** (Knowles)
- **"What mathematician would want to be expelled from the paradise which Cantor created?"** (Hilbert)

These are psychologically as well as philosophically crucial comments. At the risk of putting off older, more conservative mathematicians, who take it as a rule of faith that set theory *is* consistent, *must* be consistent, further introductory remarks will be made concerning the possibilities of "Hilbert's Expulsion", and in following sections a formal basis will be made for formally accepting this (Prophesied) Heresy: "The Destruction of The Temple". The further introduction and formal analyses that follow are intended to help overcome the almost insurmountable psychological barriers involved in communicating within the mathematical community concerning findings of actual serious fault with set theory.

In order not to be "expelled from the paradise", i.e. in order to (try to) avoid turning acceptable paradox into unacceptable inconsistency, we... we do (and fail to do) many things. We even selectively abandon fundamental principles of mathematics, such as standard rules of inference and the definition of inconsistency. One of the most common examples of such abandonment is in the transfinite arithmetic of set theory. It is the highly questionable avoidance of the otherwise standard application of the otherwise standard rule of inference that equal quantities can be subtracted from both sides of an equation in a way that maintains equality invariant to the fundamental equation of transfinite cardinal arithmetic: $\aleph_0 + 1 = \aleph_0$. The reason usually given is that "if [that derivation] was allowed, it would obviously lead to inconsistency (of an undesirable sort)". This reason given is itself "a Sign" of the abandonment of the mathematical definition of inconsistency for set theory, at least context sensitively in the transfinite portion, and of its consequences. (The Banker's Paradox suggests the term "Bankruptcy".) Set theory would not survive such inconsistency as it was never intended to be "paraconsistent". (Paraconsistent systems allow inconsistency without thereby proving all possible theorems. More on this later.) There are many more such examples.

The fundamental equation of transfinite cardinal arithmetic, $\aleph_0 + 1 = \aleph_0$, was considered paradoxical when Cantor first introduced it, since by a standard rule of inference one should perforce also derive the theorem of equality $1 = 0$. So that standard rule of inference was pseudo/quasi-formally abandoned, in a highly context sensitive fashion, and without any formal foundation being laid for its abandonment. But even though accepting $\aleph_0 + 1 = \aleph_0$ required mathematicians to (partially, intermittently and context sensitively) abandon the standard rule of inference of being able to subtract equal quantities from both sides of an equation while leaving equality invariant, mathematicians did not then find set theory to be formally inconsistent, or even otherwise unsatisfactory. This abandonment itself is paradoxical in historical, philosophical and psychological senses, but exploring that will have to wait. Formally exploring $\aleph_0 + 1 = \aleph_0$ in depth yields more paradoxes, ones that offer deeper insight into a possible resolution of the Vanishing Remainders Paradoxes and many others that indicate systemic "inadequacy".

The theorem/result that $\aleph_0 + 1 = \aleph_0$ is classically proven by means of constructing a bijection (the currently standard term for the venerable but now almost unused "one-to-one and onto function") from $\mathbb{N} \cup \{0\}$ onto $\mathbb{N}$, demonstrating that $|\mathbb{N} \cup \{0\}| = |\mathbb{N}|$ (where $|S|$ is the cardinality of the set S, and $\mathbb{N} \equiv \{1,2,3...\}$ is the set of all natural numbers, and thus the set union of $\mathbb{N}$ and $\{0\}$ has 1 more element in it than $\mathbb{N}$). (The practice of defining $\mathbb{N}$ as $\mathbb{N} \equiv \{0,1,2,3...\}$, modernly somewhat popular, should not be a problem here.) Since $|\mathbb{N} \cup \{0\}| = |\mathbb{N}| + |1| = \aleph_0 + 1$ and $|\mathbb{N} \cup \{0\}| = |\mathbb{N}| \equiv \aleph_0$, the equation $\aleph_0 + 1 = \aleph_0$ is considered to follow immediately. The bijection in question is standardly constructed using an $n \rightarrow n + 1$ mapping: for every n in $\mathbb{N} \cup \{0\}$ there is – or seems to be – a unique $n + 1$ in $\mathbb{N}$, and vice-versa. (This will all be formally analyzed in later sections.)

In the face of $\aleph_0 + 1 = \aleph_0$, since the theorem that $1 = 0$ would be considered not mere paradox but inconsistency (unacceptable since e.g. all natural numbers would become provably equal), we back off from the standard definition of a mathematical theory being *all* the theorems that can possibly be derived from the axioms *and* the rules of inference (i.e. we back off from "if you *can* derive it, you *must* derive it"), and from the standard definition of the inconsistency of a theory being the *possibility* of formally deriving both a theorem and its negation from those same axioms and rules of inference.

Relatedly, we also back off from the standard principle of always being able to replace a defined/constructed entity/symbol by its initial definition/construction. E.g. once we define/construct transfinite arithmetic from finite arithmetic, we allow proofs using a "simultaneous" mapping from n to $n + 1$ for all natural numbers n in $\mathbb{N} \equiv \{1,2,3...\}$ without any reference to the serial definition/construction of the natural numbers and thus of the mapping, and we get very different results.

- ❋ And we back off from these formal definitions and principles for the wrong reason(s): to avoid the "cognitive dissonance" of publicly confessing the inconsistency of set theory and raising serious doubts about the 2/3 of modern mathematics to which it is fundamental, not to mention all the published and about to be published papers...

❋ In fact, mathematicians should not worry about all the published and about to be published papers and other works. (The Bible has quotes about such things and burying them.) Actually, they will be quite important for studies of "just how did this all happen?" Trying to resurrect set theory, real number theory, measure theory, even logic, etc. will produce a great need for an enormous supply of papers of all sorts that will initially be quite easy to produce. There is a "sod crop" of such just waiting for the new "farmers" of the upcoming "agricultural revolutions"; they'll "spring up like weeds", for many years to come. For the first time in many decades set theory will again become a field of research, even cutting edge. So, there are no realistic worries about publish-or-perish in mathematics for the next century or so. The egg is on *everyone's* face, so no worries there, either.

Please! No jumping out of windows! Experienced department heads already know how to roll with the punch lines. Laugh and God will laugh with you! Remember, your department chairs and funding agencies, if any, can't afford to do otherwise, either. Look on the bright side: if anything, funding is likely to increase with the obviously increased need. That's what counts.

With regard to the fundamental theorem of set theory that $\aleph_0 + 1 = \aleph_0$ we have selectively, in a highly dubious context sensitive fashion, and without preparing a meta-theoretically proper theoretical foundation, deliberately failed to apply a standard rule of inference merely because it would yield a standard inconsistency (in a theory not noted for its paraconsistency; see just below). We can speak of unspoken, hidden "Rules of Deference" in such situations, where we pseudo/quasi-formally refrain from performing a standard derivation, e.g. applying a standard rule of inference, which derivation/application, by definition of a mathematical "theory", must give us a valid theorem of the theory, and thus gives us a contradiction if and only if the theory is inconsistent.

- **ESSENTIAL: *Any* mathematical theory will be "consistent" if we refuse to standardly prove/derive a theorem merely because it contradicts a previously proven/derived theorem. NOTE: some professional mathematicians have actually said things to the effect that: "The assumption that one can validly derive this (or any) contradiction is proven false by (that same) contradiction." This is merely "psychological primogeniture", and we need to recognize it as such.**

(Paraconsistent logics and inconsistent mathematics, which have become popular in recent decades, allow inconsistency to exist without then being able to prove all possible theorems – although the assumption that "deriving all possible propositions is bad" is as arbitrary as the assumption that "inconsistency is bad". So, for example, they might forgo an "Expansion Rule" which says that for any propositions P and Q, $P \supset (Q \supset P)$, i.e. if P is true, then any Q implies P; in symbolic logic it might look like $A \rightarrow A \vee B$. Once one obtains both P and *not* P, i.e. the first contradiction, it is then trivial to derive any *not* Q, and thus any proposition. Similarly for symbolic logic. The question of where this mysterious Q came from, or P for that matter, is never asked.)

- **QUICK NOTE on standard Proof by Contradiction: once known by the Latin "reductio ad absurdum", "proof by contradiction" is an ancient method of indirect proof (as opposed to direct deductive proof). One**

assumes the *falsehood* of the theorem that one intends to prove, then derives a contradiction, thus indirectly proving the theorem. (This method of proof was developed in an ancient pre-Gödel world, and suffers from implicit assumptions that obviously falter badly in a post-Gödel world. This is very important, but cannot be pursued here.) There has been and still is a problem, however, and that is that it is quite possible for the initial theory or system to be inconsistent without the new assumption (of the falsehood of the intended theorem), but unrecognized as such. (One cannot, by any usual logical means, make a standardly inconsistent system standardly consistent just by adding a new assumption.) If the initial theory is inconsistent, then it is standardly quite easy to derive any proposition (and/) or its negation, so "proving" that "the given assumption generates a contradiction" (the usual supposition in such a case) is not what it seems to be. It is quite easy to discard what could be extremely valuable theorems or postulates in this fashion, without even using the assumption or the intended theorem in the derivation of the offending contradiction.

Paradox in set theory is usually associated with the Axiom of Choice, the use of which seems to lead to rather more paradox than many mathematicians have been comfortable with (notably, Zermelo; see "Quick Remarks on the Axiom of Choice", page 48). But, in fact, unacceptable paradox in set theory will be seen to derive from the Continuum Hypothesis, or rather from the accepted fundamentals on which it was based (especially $\aleph_0 + 1 = \aleph_0$). It is the assumption (ostensibly proven, but in fact false) that transfinite sets can be bijected with proper subsets of themselves that leads visibly to the Continuum Hypothesis, and invisibly to "problematic" Overlooked Paradoxes.

Besides the Vanishing Remainders and other paradoxes seen above, and besides known paradoxes that are not yet adequately appreciated (such as the formally bypassed paradoxical theorem that $1 = 0$ that can be standardly proven/derived, and thus *theoretically must be* standardly proven/derived, from $\aleph_0 + 1 = \aleph_0$), there are other paradoxes that are as yet unrecognized in set theory, ones that are more obviously fundamental. These paradoxes, some of which will be formally analyzed in this paper, also suggest that a new kind of non-Cantorian set theory is needed for transfinite sets and their arithmetic, at least one that satisfactorily resolves all known paradoxes, especially the more embarrassing ones, and in particular the as yet Overlooked Paradoxes presented in this paper.

"Non-Cantorian set theory" has heretofore been defined (with great lack of imagination concerning the entire range of possibilities) as an otherwise standard set theory that axiomatically rejects the Continuum Hypothesis, as opposed to deriving a falsification of it. The Continuum Hypothesis has so far seemed to be an independent axiom (e.g. Gödel and Cohen). However, the Continuum Hypothesis will be found to be not only formally falsifiable, but provably false in strict set theory terms (see Theorem 12, page 35). A new set theory will be needed that more than just rejects the Continuum Hypothesis, or even falsifies it (since it is derivably false within any standard, non-paraconsistent set theory). An alternative to the Continuum Hypothesis, a "Quantinuum Hypothesis" (page 45), will be seen to derive naturally from all this, and to suggest paradigms for new theories of sets, real numbers, measure(s), etc.

The desirability of developing such a Quantinuum Hypothesis and a new and non-standardly non-Cantorian set theory (or theories) is proposed in this paper, but in the sense of necessity, not just in the sense of the explorations of new systems for their own sake, as with non-Euclidean geometries. A new theory is needed with a whole new concept of infinity or "transfinity", one from which e.g. "quantinuous" infinitesimals/-"transfinitesimals" derive naturally. An example of a possible benefit of such a new non-Cantorian theory is that it might offer a resolution to the problem of renormalization that we find in e.g. quantum mechanics, where, related to the $\aleph_0 + 1 = \aleph_0$ paradox, infinite quantities *are* subtracted – with a hope and a prayer – from both sides of various equations to get equations of finite quantities, equations that have been found to be both theoretically and pragmatically useful, even if also quite... iffy.

The keys to formally finding, exploring, analyzing and resolving these Overlooked Paradoxes relating to $\aleph_0 + 1 = \aleph_0$ lie in the bijections that ostensibly prove that result, and in certain bijectivity preserving permutations of them (of which we saw physical analog counterparts in The Good Shepherd's Paradoxes, above). We will look at these after a review of mathematical induction.

(Mild Digression concerning formality and rigor: there is always the question of how far to take "formality" and "rigor". Is there a cut-off limit on how small minutiae need to get to cease to be important? on how many minutiae of a given size need to be aired? Löwenheim-Skolem was/were mentioned above. This underappreciated theorem which bears their names, concerning models and non-standard models, can be taken to make mathematically explicit that one can never "completely" formalize a model, that there are always yet many more models, especially with higher cardinalities, with yet many many more minutiae that would be needed to formally, rigorously and precisely select the "one" model that one is interested in, so that no one can quibble about it *successfully*... now, in true Gödelian fashion, theoretically proven to be "the impossible dream". In fact, the theorem really says that with a given level of formal rigor we can only merely get a class of models, just as with a finite set of real numbers we merely get an infinite class of polynomials that has those reals as roots, and adding more roots... well, a word to the wise. So Russell and Whitehead were never really going to find their "source of the Nile" concerning "1 + 1 = 2 for average values of 1, 2, + and =". And so, any kvetching about the formal rigor offered in this paper... creatively Artful and Entertaining, please.)

A Review Of the Use of Induction, Finite and Transfinite

Mathematical induction is important in what follows, and many mathematicians are unaware of certain important ramifications of induction, both finite and transfinite. For example, many mathematicians mistake the meaning of the "finite" of "finite induction" and sometimes (with apparent "context sensitivity") think that one can only prove theorems for/about finite sets of natural numbers, or prove a proposition "for *any* natural number", but not "for *all* natural numbers". (We caught quick glimpses of this above. Another example will be offered with regard to Theorems 8 and 10, pages 30 and 32.)

And this situation has a veritable host of "psychological factors" that suggest being quite careful with regard to formal rigor and/or rigorous formality. At the risk of annoying repetition, such psychological factors can be summed up by the observation:

- ❋ **"*All* mathematicians believe it is *theoretically* possible that set theory is inconsistent. *No* mathematicians believe it is *actually* possible that set theory is inconsistent."** (Knowles)

and the historically rhetorical – but soon to be practical – question:

- ❋ **"What mathematician would want to be expelled from the paradise which Cantor created?"** (Hilbert)

So, review and clarification of mathematical induction, both finite and transfinite, are in order.

What is now considered standard finite induction comes from the postulates of Peano (Giuseppe, 1858-1932) that define the system of natural numbers. (Some like to note that the postulates originated with Dedekind.)

(Standard) Finite Induction

If it can be shown for propositions *P*(*n*), where *n* is a natural number, that:

1) ***P*(1) is true (the base clause); and that**
2) **if *P*(*n*) then *P*(*n* + 1) (the recursion clause, also written *P*(*n*) → *P*(*n* + 1))**

then it has been proven that *P*(*n*) for all natural numbers *n*.

Finite induction, which is integral to set theory, proves a countably infinite (or "denumerable") number of propositions, one for each natural number. (It is an infinite or transfinite "schema", sometimes generalized to allow starting at $n \neq 1$.)

To help counter some of the more common confusion with regard to finite induction, the following theorem is offered, which to some may seem to be equivalent to general/second kind induction, but which has a clearer statement with regard to the treatment in this paper. It is offered here for purposes of clarity and cogency, and is not considered a main result of this paper.

Theorem 1a: "Transfinite Case" Finite Induction

If it can be shown for propositions *P'*(*S*), where *S* is a set of natural numbers, that:

1) ***P'*({1}) is true (base clause);**
   and that
2) **if *P'*({1...*n*}) then *P'*({1...*n* + 1}) (recursion clause)**

then it has been proven that *P'*(ℕ), i.e. proven for the "transfinite case", as well as proven that *P'*({1...*n*}) for every natural number *n*.

We can also say that if *P'*({1...*n*}) is true "for all finite *n*", then *P'*(ℕ) is also true, again, the "transfinite case".

Proof: If we are constructing a set S, $1 \in S$ and $1...n \in S \rightarrow 1...n,n + 1 \in S$ (and no other elements in this set S) is logically equivalent to the standard definition of $\mathbb{N}$, i.e. $S \equiv \mathbb{N}$. If that isn't sufficient, consider the following: Conditions 1) and 2) construct such a set S with property P'. Condition 2) guarantees the property P' is inherited by all thus constructed supersets of $\{1\} \subseteq S$, which latter is condition 1); thus P' is inherited by $S \equiv \mathbb{N}$, therefore $P'(\mathbb{N})$. The argument that $S \neq \mathbb{N}$ fails, since in that case S must differ from $\mathbb{N}$ by some least element m (i.e. m is in $\mathbb{N}$ but not in S); m cannot be 1 by condition 1; therefore $m - 1 \in S$; by condition 2, $m - 1 + 1 \in S$, i.e. $m \in S$, which contradicts the assumption m is not in S. Thus standardly accepted mathematical reasoning gives us $S = \mathbb{N}$.

Mathematicians usually accept that one can prove inductively that a proposition $P'(\{1...n\})$ can be true "for any/all finite n", but for some strange reason they sometimes object to the necessarily consequent truth of $P'(\mathbb{N})$, even though it follows not only from the standard reasoning given above, but from standard general or second kind induction.

Thus it was felt that this review of finite induction was in order here.

We can also note that there is an important difference between "Transfinite Case" Finite Induction and standard Finite Induction: standard finite induction does not try to hold that $P(\aleph_0)$. In fact, set theory holds that $\aleph_0$ is not an element of $\mathbb{N}$, even though it is the cardinality of ℕ and every predecessor set has its maximal element as its cardinality. (For related paradoxes, see "The Maximal Element of ℕ Paradox, and The $\aleph_0$ As A Natural Number Paradox" on page 47.)

It is also worth taking a quick look at transfinite induction, a form of induction on ordinal numbers as opposed to natural numbers (which are considered to be finite cardinal numbers). It is intimately related to the Axiom of Choice and the well-ordering theorem (that any set can be well-ordered), considered to be equivalent, in fact.

(Standard) Transfinite Induction

If it can be shown for a proposition *P* that:

1) *P* holds for the first element, *α*, of a well-ordered set, *S*, i.e. *P*(*α*), and

2) if *P* holds for an element *β* whenever it holds for all predecessors (under the well-ordering) of *β*

then one may conclude that *P*(*α*) holds for all elements of *S*.

Also NOTE: transfinite induction normally works just as well for finite sets.

Finite and transfinite induction have something in common that is relevant to our situation here: if *P*(*n*) can be proven for an *arbitrary n* independently of the truth/proof of any other *P*(*m*), then the base clause and the recursion clause (in either finite or transfinite induction) *always* trivially hold, and the proof by either kind of induction is trivially valid. An extremely simple example: if we have a bijection *B* from a set *S* onto itself, and we can show that, for an arbitrary element *n* in *S*, *n* is bijected onto itself under *B*, then we have trivially shown that all the elements of the set are bijected onto themselves under *B*, and thus that *B* is an identity bijection. This is important enough here to rate a theorem, also for purposes of clarity and cogency, and also not considered a main result of this paper.

Theorem 1b: "Arbitrary Element Induction" Theorem

1) **If it can be shown that, for an arbitrary natural number *n*, a proposition *P*(*n*) is true, then, trivially equivalently, finite induction holds and it has been proven that *P*(*n*) is true for all natural numbers *n*.**
2) **If it can be shown that, for an arbitrary ordinal *α* (with possible restrictions, as mentioned above), a proposition *P*(*α*) is true, then, trivially equivalently, transfinite induction holds and it has been proven that *P*(*α*) for all ordinal numbers (with possible restrictions, as above).**
3) **In general, if it can be shown that, for an arbitrary element *x* of a set *S*, *P*(*x*) is true independently of the value of** *P* **for any other element of that set, then, trivially equivalently, it is true that *P*(*x*) for all elements (each and every element) *x* in *S*.**

Proof: 1) If *P*(*n*) is true for an arbitrary natural number *n*, independently of the value of *P* for any other natural number, then the base clause and recursion clauses trivially hold, and a standard proof by finite induction trivially holds as well. The other half of proof of equivalence is likewise trivial. 2) If *P*(*α*) is true for an arbitrary ordinal *α* (possibly with some restrictions), independently of the value of *P* for any other ordinal, then the combination base and recursion clause trivially holds, and a standard proof by transfinite induction trivially holds as well. The other half of proof of equivalence is likewise trivial. 3) The third case is stated here with no further proof.

It is surprising how often seemingly trivial theorems and other results can be useful.

Bijectivity Preserving Permutations of Bijections

Bijections are now the standard method for counting, i.e. if the elements of one set can be put into a 1-to-1 correspondence with the elements of another set, this demonstrates that the sets have the same "cardinality" or "number of elements". Set theory and its transfinite arithmetic are based on these bijections, some of which are "paradoxical" (using the term here in its historically usual informal sense), as when a transfinite set is bijected with a proper subset of itself. Peirce (Charles Sanders, 1839-1914), and later (but much more famously) Dedekind (Julius Wilhelm Richard, 1831-1916), suggested that the existence of a bijection of a set with a proper subset of itself was in fact definitional of transfinite sets.

Transfinite arithmetic depends completely on such "paradoxical bijections". Such bijections are used to prove standard theorems that $n\aleph_0 = \aleph_0$, $n2^{\aleph_0} = 2^{\aleph_0}$, etc. For $n = 2$, we see a set theoretical counterpart to the Banach-Tarski Paradox of the (currently) standardly accepted paradoxical measure theory (rigidly moving a small number of pieces of a solid sphere and reassembling them to form 2 solid spheres with the same size as the original sphere). The "paradoxical bijections" that will be formally defined in the next section are another set theoretical counterpart, differing in that they demonstrate the ultimate foundation of these paradoxes, "the source of the Nile".

Cantor's concepts of "abstraction with regard to order" and its handmaiden, "reordering" (of the bijective sub-mappings in a bijection; this is, still unrecognizedly, essentially related to Banach-Tarski), come into this heavily, being used to standardly prove that bijections can be permuted by changing the "order" of the 1-to-1 sub-mappings so as to allow further elements to enter into one side of the bijection while still maintaining the global bijectivity invariant. This is now all accepted as standard in all modern variants of Cantor's set theory, however axiomatic... however naïve.

Perhaps the most paradigmatically fundamental such "paradoxical bijection" – proposed here as a good formal term for such, and formally defined later – is the standardly accepted "bijection" from $\mathbb{N} \cup \{0\}$ onto $\mathbb{N}$, often used to prove that $\aleph_0 + 1 = \aleph_0$. One reason such bijections seem paradoxical is that the sets are held to have "the same number of elements", i.e. "the same cardinality", even though standard set subtraction yields a non-empty set. Another reason, the paradoxical nature of the intimately related $\aleph_0 + 1 = \aleph_0$, has already been commented on (e.g. page 19).

But another Paradox has been Overlooked. It is quite possible to derive, from the standard axioms and rules of inference of set theory, a counterpart to set subtraction for bijections, specifically for these paradoxical "bijections" from sets onto proper subsets of themselves that are so fundamental to transfinite counting and arithmetic. Set subtraction gives us quite a different picture of the apparent cardinalities of $\mathbb{N} \cup \{0\}$ and $\mathbb{N}$ than does the standard bijection from $\mathbb{N} \cup \{0\}$ onto $\mathbb{N}$. The set difference gives us the set $\{0\}$, with one element in it, thus suggesting that the *number* of elements $\mathbb{N} \cup \{0\}$ is *not* equal to, but instead *strictly greater* than the *number* of elements in $\mathbb{N}$, in fact, 1 greater. After all, any single 1-to-1 correspondence between elements should be abstractly equivalent to any other, should it not?

It turns out that there is a formal counterpart to set subtraction in certain bijectivity preserving permutations of bijections. When applied to bijections from sets onto proper subsets of themselves, bijections that have been considered *acceptably* paradoxical as well as formally valid since the days of Cantor, these permutations, that trivially maintain bijectivity invariant, show that there *must* also exist bijections from non-empty sets onto the empty set, $\phi \equiv \{\ \}$. Such "bijections" are especially paradoxical, probably too paradoxical.

- **If only precisely 1 element in the pre-image set, an element common to both sets of a bijection, can be bijected onto precisely 1 element in the image set, that other 1 element might as well be itself, its identity image. But in that case it-paired-1-to-1-with-itself can be subtracted out of both sets without adversely affecting the bijectivity of the remaining subbijection from the set of the remaining pre-image elements onto the set of the remaining image elements. But what remains when all the common elements have been subtracted out of a bijection between a set and a proper subset of itself? A "Paradoxical Bijection"...**

In the next section these "Paradoxical Bijections" will be shown to be members of a formally definable class of such, much like "paradoxical measure" has been formally defined. (In the last few decades paradoxical measure has become both a popular and a standard field of study.) An important class of paradoxes relating to such Paradoxical Bijections has been named The Bijection Permutation Paradoxes. The Bijection Permutation Paradoxes 1 (page 36) and 2 (this and next section; page 34) are formal counterparts of The Good Shepherd's Paradoxes, i.e. of The Golfer's Paradox (page 13) and The Banker's Paradox (page 16), respectively. But they are here presented in reverse order for convenience.

We will forgo completeness in favor of adequacy, and try to restrict definitions, theorems, proofs, and observations to those that are critical or potentially paradigmatic. (E.g. the definition of equality of bijections is left "intuitively obvious".) The preparatory definitions, theorems, and notational usages are obvious enough that the reader may wish to proceed quickly to Theorem 9 (page 32).

Notation 1: Sets of Pre-Image Elements, etc.

"*SP*" will generally be used to denote a set of pre-image elements of a bijection *B*, "*SI*" to denote a set of image elements of *B*, and "*SC*" the set of all elements common to both *SP* and *SI*. *S'* will often denote a subset of a set *S*. "*EP*" will generally be used to denote a pre-image element of (e.g. of a pre-image set "*SP*" of) *B*, "*EI*" to denote an image element of *B*.

Definition 1: Disjoint Bijections

2 bijections, *B1* from set *SP1* onto set *SI1* and *B2* from set *SP2* onto set *SI2*, are "disjoint bijections" if (and only if) *SP1* and *SP2* are disjoint, and *SI1* and *SI2* are disjoint.

Definition 2: Subbijection and Superbijection

A bijection *B'* from set *SP'* onto set *SI'* is a (proper) "subbijection" of a bijection *B* from set *SP* onto set *SI* if (and only if) *SP'* is a (proper) subset of *SP*, *SI'* is a (proper) subset of *SI*, and every element of *SP'* has the same image (in *SI'*) under *B'* as it has (in *SI*) under *B*. The bijection *B* is a (proper) "superbijection" of its (proper) subbijection *B'*.

Notation 2: Subbijections, etc.

_B'_ will often be used to denote a subbijection of a bijection *B*, or, distinctly, a permutation of *B*. If *B* is a bijection from a set *SP* of pre-image elements onto a set *SI* of image elements, one can write $B(SP) = SI$, $B(SP \rightarrow SI)$ or $B(SI \leftarrow SP)$ to indicate the bijection and the sets/subsets it is being applied to. I.e. if *SP'* is a subset of *SP*, one can also write $B(SP') $ to indicate the subset *SI'* of *SI* that is the image of *SP'* under *B*; i.e. one can write $B(SP') = SI'$. Similarly, if *EP* is an element of *SP*, one can write $B(EP)$ to indicate the

image *EI* in *SI* of *EP* under *B*; i.e. one can write *B*(*EP*) = *EI*. One can also write *B*(*EP* → *EI*) or *B*(*EI* ← *EP*) to indicate the bijection and particular element(s) that it maps bijectively. With regard to bijective inverses of sets, elements, or bijections, if e.g. *B*(*SP*) = *SI* or *B*(*EP* → *EI*), one can also write B^{-1}(*SI*) (= *SP*), B^{-1}(*SI*→ *SP*), B^{-1}(*SP*← *SI*), B^{-1}(*EI*) (= *EP*), B^{-1}(*EI* → *EP*), B^{-1}(*EP*← *EI*), or any other reasonable variant to indicate such.

Definition 3: Union of Bijections

If we have 2 bijections, *B1* from *SP1* onto *SI1* and *B2* from *SP2* onto *SI2*, the "union of the bijections", *B1* ∪ *B2*, is the (not necessarily bijective) mapping *M* from *SP1* ∪ *SP2* onto *SI1* ∪ *SI2* in which the image of every element of *SP1* ∪ *SP2* is the same under *M* as under *B1*, *B2*, or both if either of the sets *SP1* ∩ *SP2* or *SI1* ∩ *SI2* is non-empty.

Notation 3: Unions of Bijections

For (*B* =) *B1* ∪ *B2* we can also write (*B* =) *B1* + *B2*.

We will here forgo the usual theorems about commutivity, associativity, etc.

Definition 4: Partition of a Bijection

If we have a bijection *B* from *SP* onto *SI*, a (proper) partition of *B* is a collection of disjoint (proper) subbijections of *B* such that their union is *B*.

Theorem 2: Subsets Define Subbijections and Subbijections Define Subsets

If we have a bijection *B* from *SP* onto *SI*, then any (proper) subset *SP'* of *SP* or *SI'* of *SI* defines a (proper) subbijection *B*' of *B*. Likewise, any (proper) subbijection *B'* of *B* defines (proper) subsets *SP'* of *SP* and *SI'* of *SI*. This result extends to partitions of bijections or their (pre-) image subsets.

Proof: Any *SP'* that is a (proper) subset of *SP* has an image *SI'* under *B* that is a (proper) subset of *SI*. Let *B'* be the bijection from *SP'* onto *SI'* in which every element's image under *B'* is the same as under *B*. *B'* is the required subset-defined subbijection of *B* defined by *SP'*. Likewise for any subset *SI'* of *SI*. Subbijections define subsets similarly, by definition. Extension to partitions is trivial.

Theorem 3: Bijectivity Preserving Union of Bijections

The union of 2 bijections, *B1* from *SP1* onto *SI1* and *B2* from *SP2* onto *SI2*, is itself a bijection if and only if one or more of the following:

1) *B1* and *B2* are disjoint bijections,

2) *B1* and *B2* are both subbijections of a third bijection,

3) or in general the subbijection of *B1* from *SP1* ∩ *SP2* onto *SI2* ∩ *SI2* is (identically) equal to the subbijection of *B2* from *SP1* ∩ *SP2* onto *SI1* ∩ *SI2*. Stated without proof.

Theorem 4: Subtraction of a Subbijection from a Bijection, and Subbijection Partition (-ing) of a Bijection

Any (proper) subbijection *B'* from *SP'* onto *SI'* of a bijection *B* from *SP* onto *SI* can be subtracted from *B* yielding the disjoint (proper) subbijection *B''* of *B* from *SP* – *SP'* onto *SI* – *SI'*. Since *B'* and *B''* are disjoint and their union is *B*, together they partition *B*. Stated without proof.

Notation 4: Subtraction of Bijections.

In the case that *B1* and *B2* are disjoint bijections and their union is *B* = *B1* + *B2*, we can also write (*B2* =) *B* – *B1* to indicate the subtraction of *B1* from their (common) superbijection *B*.

The above are pretty much just straightforward counterparts of similar definitions and theorems for sets. The following are specific to bijections.

Notation 5: “Links”, etc.

The bijective mapping from one element onto another can be referred to as a “link”, and one can refer to “switching links” to mean “switching elements” so that elements of the pre-image set are bijectively mapped onto different elements of the image set.

Clarification 1: “Permutation”.

The use of the term “permutation” may here be somewhat ambiguous, as it often is in mathematics. It may be used to mean on the one hand the “transformational change” that is applied to and/or occurs in e.g. a bijection, on the other the “transformed bijection”, or both, e.g. when it occurs in the context of a “sequence of permutations”. The context should make the distinction clear, if in fact it is important.

Definition 5: *n*-Element (Pre-) Image Swapping Permutation of a Bijection

An “*n*-element (pre-) image swapping (or switching) permutation of a bijection” *B* from *SP* onto *SI* ($|SP| = |SI| \geq n$) is a bijection *B’* from *SP* onto *SI* such that the images in *SI* of precisely *n* elements of *SP* (or pre-images in *SP* of precisely *n* elements of *SI*) are different under *B’* than they are under *B*. An “identity image swapping permutation of a bijection” is a 0-element (pre-) image swapping permutation of a bijection, i.e. involving the swapping of precisely $n = 0$ elements, and thus the permuted bijection *B’* is identically equal to the original bijection *B*. As long as the sets of the initial bijection have no orderings that can be adversely affected (the standard definition of “bijection” takes no notice of any orderings of its sets, which by the standard definition of sets are unordered), image swapping can be equivalently thought of as pre-image swapping.

Notation 6: Ordering of Elements for Indicating Links.

If order is notationally important (e.g. to indicate specific links), it can be specified in context; e.g. if we have:

$B(EP1) = EI2$ **and** $B(EP2) = EI1$,

we can write:

$B(\{EP1,EP2\}) = \{EI2,EI1\}$,
$B(\{EP1,EP2\}) \rightarrow \{EI2,EI1\}$,
$B(\{EP1,EP2\} \rightarrow \{EI2,EI1\})$,
$B(EP1 \rightarrow EI2, EP2 \rightarrow EI1)$,

or any other reasonable variant to indicate the respective order of the bijective mappings, and preferably making it clearly explicit that such orderings are involved.

Theorem 5: *n*-Element (Pre-) Image Swapping Permutation of a Bijection

If we have a bijection *B* from *SP* onto *SI* ($|SP| = |SI| \geq \mathrm{n}$), there exists a bijection *B’* from *SP* onto *SI* such that the images in *SI* of each of precisely *n* arbitrarily chosen elements of *SP* (or pre-images in *SP* of precisely *n* elements of *SI*) are different under *B’* than they are under *B*; i.e. these elements are “swapped” under *B’*, as per Definition 5. The smallest number of (pre-) image elements that can be thus swapped (or of links that can be permuted) in a non-identity (pre-) image swapping permutation of *B* such that the resulting mapping *B’* remains a bijection from *SP* onto *SI* is $n = 2$.

Proof: We will limit the explicit proof to *n* = 2. Let *i* and *j* be any 2 distinct elements of *SP*. Along with their images in *SI* under *B*, they form a subbijection of *B*, *B''*, from {*i*,*j*} onto *B*({*i*,*j*}) = {*B*(*i*),*B*(*j*)}. (The sets here are notationally ordered, per Notation 6.) The bijection *B''* partitions *B* into *B''* and *B'''* = *B* – *B''*. If we permute *B''* into *B''''* by swapping (*i* and *j* or) the images of *i* and *j* under *B''*, i.e. making *B''''*(*i*) = *B''*(*j*) and making *B''''*(*j*) = *B''*(*i*), *B''''* is now a bijection from {*i*,*j*} onto {*B*(*j*),*B*(*i*)} instead of onto {*B*(*i*),*B*(*j*)}. The subbijection *B'''* = *B* – *B''* of *B* remains unpermuted by this element swapping permutation of *B''* into *B''''* (and thus becomes/is an unpermuted subbijection of *B''''*). The union of *B'''* and *B''''* is a bijection from *SP* onto *SI*, per Theorem 3. The bijection formed by this union *B'* = *B'''* + *B''''* is the required bijection *B'* of the first part of the theorem. That this *n*-element (pre) image swapping permutation can be likewise done for n > 2 and the second part of the theorem, that *n* = 2 is the smallest (non-zero) number of elements that can be (non-trivially) swapped or switched, are obvious enough that they can be stated here without proof.

This last definition and theorem, though seemingly trivial, are given because they are paradigmatic both for general permutations of bijections, and eventually for a new set theory. A simple but paradigmatic example of their use suggests itself.

Theorem 6: Single Common Element Pairing Permutation of a Bijection

If we have an arbitrary bijection B from *SP* onto *SI*, and *SP* and *SI* have a non-empty set *SC* of elements in common, and we have an element *n* arbitrarily chosen from *SC*, and this *n* in *SP* (and *SC*) is not bijectively mapped onto itself in *SI* (and *SC*) under *B* (i.e. *B*(*n*) = *j* ≠ *n* for some *j* in *SI* and *B*(*i*) = *n* ≠ *i* for some *i* in *SP*), then, per Definition 5 and Theorem 5, there exists a bijection *B'* from *SP* onto *SI* such that the 2 pre-image elements *n* and *i* in *SP* of each of 2 and only 2 image elements *n* and *j* of *SI* are different under *B'* than they are under *B*; i.e. $B'(n) = B^{-1}(n) = n$, $B'(i) = j$, and $B^{-1}(j) = i$; i.e. these (pre-) image elements are "swapped" so that *n* is bijectively mapped onto itself under *B'*. If *n* is initially already bijected onto itself under *B*, we can alternatively speak of the permutation as an identity (single common element pairing) permutation/swap. If any common element *k* in *SP* is already bijected under *B* onto itself in *SI*, then, after this permutation of *B* to *B'*, this *k* will remain bijected onto itself under *B'*; this gives us idempotence for this class of permutations as operators or functions.

Proof: By Theorem 5 we can swap the pre-images in *SP* of 2 arbitrarily chosen elements of *SI*. Let one of those arbitrarily chosen elements of *SI* be *n* (also in *SC*), the pre-image (counterimage or inverse image) of which is *i* in *SP*, and let the other be the image *j* in *SI* of *n* in *SP*. After applying Theorem 5, *n* in *SP* will be the pre-image of *n* in *SI*, (i.e. *B*(*n* → *n*)) and *i* in *SP* will be the pre-image of *j* in *SI* (i.e. *B*(*i* → *j*)). Any element *k* already bijected onto itself cannot be bijected either from *n* or *i* in *SP*, or onto *n* or *j* in *SI* (unless *k* = *n* in an identity single common element permutation), so any *k* onto *k* subbijection will remain invariant under this permutation of *B*, as will every other subbijection not involving *n* or *i* in *SP* or *n* or *j* in *SI* (also as per Theorem 5).

Let *SCP*(*n*,*b*) be a class of Single Common Element Pairing Permutations or permutation operators or functions that can be applied to an arbitrary element *n* and an arbitrary bijection *b*. The permutation *SCP*(*n*,*b*) can be applied to an arbitrary bijection *B* giving us a bijection *SCP*(*n*,*B*) = *B'* such that:

1) if the element n is common to both sets SP and SI but n is not already bijectively mapped onto itself under B (i.e. $B(n) \neq n$), then applying the $SCP(n,b)$ operator to B will give us $SCP(n,B) = B' \neq B$ and $B'(n) = n$;
2) if the element n is common to both sets SP and SI but n is already bijectively mapped onto itself under B (i.e. $B(n) = n$), then applying the $SCP(n,b)$ operator to B gives us an identity permutation, $SCP(n,B) = B$;
3) if n is not an element common to both sets of the bijection B (perhaps belonging to neither SP nor SI), then applying the $SCP(n,b)$ operator to B gives us an identity permutation, $SCP(n,B) = B$;
4) if we let B^* be the current cumulative permutation of the initial bijection B and PN be the set of elements n such that the operator $SCP(n,b)$ has been applied at least once in the cumulative sequence of permutations of the initial B leading to B^*, any further application of $SCP(n,b)$ to B^* such that $n \in PN$ will always give us an identity $SCP(n,B^*) = B^*$. This last is an interesting generalization of idempotence for operators. This idempotence is essential to the convergence of sequences of these permutations. This is one of the reasons that these Single Common Element Pairing Permutations are paradigmatically essential for studying bijections.

This last theorem is deceptively simple. It suggests an extremely simple but paradigmatic example of its use.

Theorem 7: Permutation of a Bijection From a Set Onto Itself Into the Identity Bijection By Means of Single Common Element Pairing Permutations
If we have an arbitrary bijection *B* from *SP* onto *SI* ≡ *SP*, it can be permuted in a bijectivity preserving fashion into the identity bijection (from itself onto itself) by swapping at most 2 pre-image elements at a time, with at most one non-identity swap or permutation per element.

Proof: since the Single Common Element Pairing Permutation of Theorem 6 can be applied successfully for every element of $SP \equiv SI$ independently of each and every other element, then by Theorem 1b the theorem is proven. All elements of the set SP and set $SI \equiv SP$ will have been bijected onto themselves, maintaining invariant the bijective validity of the initial bijection in every successive permutation up to and including the final cumulatively permuted bijection. This is a trivial proof by mathematical induction (either finite or transfinite, depending on the cardinality of $SP \equiv SI$), as per Theorem 1b. Since each element, once bijected onto itself, remains invariantly so bijected, only 1 permutation or swap is needed per element (because of the idempotence referred to in Theorem 6).

This last theorem seems highly intuitively obvious, and it is. It is vaguely a variant of a software bubble sort algorithm. But for some mathematicians it ceases to be obvious when it is noticed that it does not depend on the bijection being from a set onto itself, but only on the sets having elements in common, such as when a transfinite set is bijected with a proper subset of itself. Theorem 8 is generalized so as to allow application to such situations.

Theorem 8: Permutation of a Bijection So That an Identity Subbijection Is Constructed From the Subset of Elements Common to Both Sets Onto Itself By Means of Single Common Element Pairing Permutations
If we have an arbitrary bijection *B* from a set *SP* onto a set *SI* with a set of elements *SC* in common, *B* can be permuted in a bijectivity preserving fashion, by swapping only 2 (pre-) image elements at a time, so that its

cumulative permutation remains a bijection from *SP* onto *SI* and has, as a subbijection, *B'*, the identity bijection from the subset *SC* of *SP* onto the subset *SC* of *SI*. At most one (non-identity) swap or permutation is needed per common element.

Proof: as in Theorem 7, since the Single Common Element Pairing Permutation of Theorem 6 can be applied successfully for every element of *SC* independent of each and every other element of *SC*, then by Theorem 1b the theorem is proven. All elements common to both sets *SP* and *SI* will have become bijected onto themselves, maintaining invariant the bijective validity of the initial bijection in every successive permutation up to and including the final cumulatively permuted bijection. As in Theorem 7, since each element, once bijected onto itself, remains invariantly so bijected, only 1 permutation or swap is needed per element (because of the idempotence referred to in Theorem 6).

This last theorem is clearly a generalized and formalized variant of The Banker's Paradox. It is really all we need for the crucially important further result obtained by applying it to sets ostensibly bijected onto proper subsets of themselves, but the slightly different approach that will be taken in Theorems 9-10 seems more compelling.

Along with Theorem 10, this (bijections with proper subsets) is where some mathematicians have tried to object that the theorem can only be proven for finite numbers of common elements and/or that the sequences of permutations of some bijections do not "converge" if the original bijection is thus permuted for a transfinite number of common elements. But such convergence is absolutely required in these cases, and Theorems 1a-b clearly prove the fallacy of this objection.

(Since "non-convergence" is likely to be a common objection, this issue should be clarified. The non-convergence objection is really the objection that the sequence does *not* converge to a *valid* bijection. But it is theoretically necessary that it *should* so converge. In our case here we see it *must* converge and *does* (the idempotence of Theorem 6 ensures this), but to a globally "invalid bijection". I.e. if we start with an *assumed* "bijection" from a set onto a proper subset of itself, we converge in a sequence of permutations that maintain bijectivity completely invariant to a "bijection" with one trivially valid identity subbijection where all common elements are bijectively mapped onto themselves, and – the reason why the sequence doesn't *seem* to "converge" – a second obviously invalid "subbijection" from a non-empty subset onto the empty set. This is in fact convergence, but just not to a *globally* valid bijection. The fact that the final "bijection" is globally invalid does not mean non-convergence; it means that the original assumption of an initially valid bijection is false. There is an *assumption* (ostensibly proven, which makes the problem even more problematic) in set theory that sets can be validly bijected onto proper subsets of themselves, but the "proof" of that is in fact fatally flawed (see "The Bijection Permutation Paradox 1, 'Counting' and Cantorian 'Reordering'", page 36). This convergence to a globally invalid bijection is in fact a standardly valid proof by contradiction that set theory is inconsistent. But the arguments will be made even more compelling in Theorems 9-14, pages 32-39.)

There are many other important such theorems, but the topic of permutations of bijections, paradigmatic in general, is extremely complex. We will forgo further general development along this particular line at this time.

Now we get to the most compelling of the essential results, developed along a slightly different line: subtracting the identity subbijections of (identical) paired elements as they are re-paired in a process that maintains bijectivity invariant (while guaranteeing that no element will be deprived of a bijective mapping), or are found to be already paired.

Theorem 9: Single Common Element Subtraction Permutation of a Bijection

If there exists a bijection *B* from a set *SP* onto a set *SI*, where *SP* and *SI* have an element *n* in common, then there must also exist a bijection *B'* from *SP* – {*n*} onto *SI* – {*n*}. If any *k* is not a member of either *SP* or *SI* (perhaps by having been subtracted out earlier), then, after this permutation of *B* to *B'*, *k* will *not* have become a member of either *SP* – {*n*} or *SI* – {*n*} under *B'*; for this class of permutations as operators or functions, this gives us a subtle but obvious variant of the idempotence of the Single Common Element Pairing Permutation of a Bijection of Theorem 6.

Proof: because *B* is a bijection, there are only 2 cases:

1) if *n* is already bijected onto itself under *B*, the result follows trivially since the subbijection $Bn = B(\{n\} \rightarrow \{n\})$ of *B* partitions *B* as per Theorem 4, giving us the required bijection $B' = B - Bn$ from $SP - \{n\}$ onto $SI - \{n\}$.

2) otherwise, we can apply the Single Common Element Pairing Permutation of Theorem 6, choosing *n* of *SP* as the arbitrary common element; when we do this, we have a new permuted bijection $SCP(n,B) = B' \neq B$ from set *SP* onto set *SI* with *n* bijected onto itself, and we can proceed as in part 1.

To be safe, though, we will again go onto the critical details: there exists a subbijection *B''* (of *B*) from {*n*,*i*} onto {*j*,*n*} (here both subsets notationally ordered), for some *n* and some *i* in *SP* and some *j* and *n* in *SI*, that partitions *B* into *B''* and another subbijection *B'''* (improper for $|SP| \leq 2$); *B''* can be permuted into 2 disjoint bijections (by applying Theorem 5, followed by Theorem 2 with {*n*} as the subset), *Bn* from {*n*} onto {*n*} and *Bij* from {*i*} onto {*j*} (thus trivially preserving bijectivity); if we take the union of the disjoint bijections *Bij* and *B'''*, we get the required bijection $B' = Bij + B'''$ from $SP - \{n\}$ onto $SI - \{n\}$.

The comments in the Proof to Theorem 6 (page 29) concerning the class *SCP*(*n*,*b*) of permutation operators/functions do not need to be repeated here in detail. Since permutations in this new class (here we can refer to it as *SCS*(*n*,*b*)) can only subtract common elements, it is obvious that they cannot introduce/reintroduce elements into $SP - \{n\}$ or $SI - \{n\}$. This idempotence is essential to convergence of sequences of these permutations. This is one of the reasons that these Single Common Element Subtraction Permutations are paradigmatically essential for studying bijections.

Theorem 10: All Common Elements Subtraction Permutation of a Bijection

If there exists a bijection *B* from a set *SP* onto a set *SI*, where *SP* and *SI* have a set $SC = SP \cap SI$ of elements in common, then there must also exist a bijection *B'* from $SP - SC$ onto $SI - SC$. (This can be generalized to any subset of *SC*, the set of all common elements.)

Proof: the Single Common Element Subtraction Permutation of a Bijection from Theorem 9 can be applied to *B* for every element of *SC*, in any order, yielding the required bijection *B'* from $SP - SC$ onto $SI - SC$. (The order will determine details of which elements are linked in the final bijection, but will not affect the invariance of bijectivity that is maintained throughout the sequence of permutations, one for each element of *SC*.) The theorem is trivially proven if $SC = \phi \equiv \{\ \}$, i.e. if *SC* is the empty set.

Along with Theorem 8, this is where some mathematicians have tried to object that the theorem can only be proven for finite numbers of common elements and/or that the sequence of permutations may not "converge" for some bijections if the bijection is permuted for a transfinite number of common elements. But again, in our case here, such convergence is absolutely required, and Theorems 1a-b clearly prove the fallacy of this objection. The earlier comment that the sequence of permutations converges, but to an obviously invalid bijection still holds.

NOTE the important similarity between Theorem 10 and Theorem 8 with regard to standardly accepted "bijections" from sets onto proper subsets of themselves. But also note that this second approach seems more compelling since the common elements have not merely been re-paired, but subtracted out of the picture, as they are in standard set subtraction. Every permutation re-pairs and removes a single common element identity subbijection from the cumulatively permuted original bijection, always maintaining bijectivity invariant, never depriving any element of a bijective mapping (including the subtracted identity subbijected/paired elements). Since this removed subbijection is necessarily 1-to-1, its removal is unable to invalidate either the bijectivity of the remaining sets in the permuted bijection, or the equality of their cardinalities.

As often happens in mathematics, though it seems trivially obvious, it is essential to observe and keep in mind the basics:

- **Because of the precise 1-to-1 nature of bijections:**
  1) **partitioning a bijection (by definition into disjoint subbijections) maintains bijectivity invariant;**
  2) **taking the union of disjoint bijections maintains bijectivity invariant;**
  3) **permuting a bijection by taking 2 (generalizable to *n*) of the elements in the image (or onto) set and switching their pre-images in the pre-image (or from) set (or vice-versa) maintains bijectivity invariant; (a standard theory of such permutations is still lacking, which is not surprising since it quickly leads to Theorems 6-10, and then to Theorems 11-12;)**
  4) **subtracting out a subbijection maintains bijectivity invariant, especially obviously if that subbijection is from a single common element onto its identity counterpart; that is as obviously 1-to-1 as it is possible to get in mathematics.**

Taken together these guarantee the validity of the proof of Theorem 9, and thus the proof of Theorem 10, etc.

The Bijection Permutation Paradox 2, Paradoxical Bijections, the Continuum Hypothesis, and the Inconsistency of Set Theory

The paradoxical but standard bijection from $\mathbb{N} \cup \{0\}$ onto $\mathbb{N}$ (so paradigmatically fundamental in Cantorian set theory) is usually proven to exist roughly as follows: since for every n in $\mathbb{N} \cup \{0\}$ there (apparently) exists a unique corresponding $n + 1$ in $\mathbb{N}$, and vice versa, this (ostensibly) gives us a bijection between them. This $n \to n + 1$ mapping is an example of Cantor's concepts of "abstraction with regard to order" and "reordering" that he held demonstrate that these 2 sets are of the same "cardinality". This concept of "reordering" pervades the "bijections" – the modern term for the earlier "1-to-1 and onto functions" – that underpin Cantorian transfinite cardinal arithmetic.

The cardinality of $\mathbb{N} \cup \{0\}$ is $\aleph_0 + 1$, since it has 1 more element than $\mathbb{N} \equiv \{1,2,3...\}$, and the cardinality of $\mathbb{N}$ is $\aleph_0$ by definition. Together with the standard bijection from one onto the other, and therefore the equality of their cardinalities, they give us the paradoxical yet fundamental theorem for standard transfinite cardinal arithmetic: $\aleph_0 + 1 = \aleph_0$. Remember, this fundamental theorem, *and by implication* $\aleph_0 + 1 \ngtr \aleph_0$ since we have the implicit assumption of the consistency of set theory along with its lack of paraconsistency (not to mention the theoretical exclusiveness of "=" and ">", and related logical principles), are essential to the Continuum Hypothesis, though not strictly the other way round.

But Cantor et al didn't explore far enough. The informal concept of a "Paradoxical Bijection" can be made formal, similarly to the concept of "Paradoxical Measure" in the now standardly accepted, recently popular branch of paradoxical measure theory, but with quite a different overall result. (The Banach-Tarski Paradox of paradoxical measure theory shows how it is possible to take a solid ball or sphere, divide it up into a small number of pieces, then rigidly move them to form 2 solid balls with the same size and volume as the original. This obviously relates to set theory's $2\aleph_0 = \aleph_0$ and $2 \cdot 2^{\aleph_0} = 2^{\aleph_0}$, but as yet unappreciatedly. The Bijection Permutation Paradoxes may shed some light on this.)

Definition 6: Paradoxical Bijection

A bijection between sets of provably different cardinalities, or a bijection that can be permuted in a bijectivity preserving manner to a bijection between sets of provably different cardinalities.

Theorem 11: The Bijection Permutation Paradox 2 and Paradoxical Bijections

If there exists a bijection from $\mathbb{N} \cup \{0\}$ onto $\mathbb{N}$, it is a Paradoxical Bijection since it can be permuted, in a manner that maintains bijectivity invariant (and does not deprive any element of a bijective mapping), into a bijection from $\{0\}$ onto the empty set, $\phi \equiv \{\ \}$. In general, if a bijection exists from a set S onto a proper subset S' of itself, it is a Paradoxical Bijection since it can be derived as above that there must also exist a bijection from the non-empty set difference *S – S'* onto the empty set, $\phi \equiv \{\ \}$.

Proof: it suffices to apply Theorem 10 to the standard bijection from $\mathbb{N} \cup \{0\}$ onto $\mathbb{N}$, or from any set onto a proper subset of itself. Note that it would also be easy to use Theorem 8 to achieve this same result, one of the reasons that Theorems 6-8 are paradigmatic.

Although it should not be considered "mere paradox", as most of set theory's paradoxes – mostly paradoxes of infinity – have been considered so far:

- **The Bijection Permutation Paradox 2 (which corresponds to The Banker's Paradox, page 16), of which Theorem 11 gives a paradigmatic example, should be considered a new paradox in set theory, overlooked since the days of Cantor.** (See Bijection Permutation Paradox 1, page 39.)

This paradox differs in two ways from other paradoxes: secondly, it points in the direction of resolution, in this case less the resolution of the paradox itself, and more of the system that gives rise to it, but firstly, it seems rather obviously *too* paradoxical, pointing out the *need* for such resolutions. As might be expected, there are more paradoxes related to this one (as are the Vanishing Remainders Paradoxes), and some of them will be examined in sections that follow.

Theorem 12: Continuum Hypothesis Paradox 2: $\aleph_1 = \aleph_0 + 1 > \aleph_0$.
There does not exist a valid bijection from ℕ ∪ {0} onto ℕ, and necessarily $\aleph_0 + 1 > \aleph_0$, giving us $\aleph_1 = \aleph_0 + 1$, a new and Paradoxical resolution of the Continuum Hypothesis question. Since Theorem 10 applies to sets of any cardinality, finite or transfinite, this result is general, especially paradoxically applying even to Cantor's absolutely infinite set(s) and its (their) cardinality. In particular, one does not need to resort to power sets to increase transfinite cardinalities; simple succession by adding 1 (new element) will suffice. (There are obvious generalizations of this for measure theory, integration theory, etc.)

Proof: this result follows from Theorems 10-11, especially if we wish to avoid the formally provable existence of bijections from {0} – and from many other non-empty sets – onto the empty set. I.e. we use the contradiction found by Theorems 10-11 to "prove by contradiction" that the *assumption* that there exists a bijection from ℕ ∪ {0} onto ℕ is false.

Remember, it has been shown in a very clear and cogent manner in Theorem 10 that *if* a bijection exists from ℕ ∪ {0} onto ℕ, *then* there must also exist a bijection from a non-empty set onto the empty set, a contradiction that not even making set theory paraconsistent is likely to be able to handle. The $n \to n + 1$ mapping that is usually used to construct the "bijection" from ℕ ∪ {0} onto ℕ will be analyzed in the next section, "The Bijection Permutation Paradox 1, 'Counting' and Cantorian 'Reordering'", page 36, and the flaws whose existence is demonstrated by the above results (including The Good Shepherd's Paradoxes) will be confirmed.

This result is fundamentally paradigmatic for a new (non-standardly non-Cantorian) set theory and any theory that will use it as a foundation. Along with Theorems 9-11, it shows that one needs a very different kind of foundation than set theory can supply to allow any kind of "infinity + 1 = infinity" result. That result is like "ceiling arithmetic" instead of standard arithmetic. We need a "standard", "Euclidean" set theory as a sound basis for branching out into "non-Euclidean" set theories.

These bijection-cardinality results match up perfectly with standard set subtraction (e.g. for ℕ ∪ {0} – ℕ), since in set subtraction each common element automatically gets paired up with precisely itself as it gets subtracted (in Theorems 10-11 it first gets "de-linked" from whatever precisely 1 element it is bijected onto), whereas the standard formal existence of a bijection from ℕ ∪ {0} onto ℕ and thus their having the same cardinality do not match up with set subtraction at all. They also make clear how replacing the standard $\aleph_0 + 1 = \aleph_0$ with $\aleph_0 + 1 > \aleph_0$ could eventually lead to a satisfactory resolution of the renormalization problem that plagues physics, notably quantum mechanics, and also of the Banach-Tarski Paradox and Paradoxical Measure of paradoxical measure theory (a new branch of measure theory, popular in recent decades and now considered standard).

They also give us the:

Continuum Hypothesis Paradoxes 3: The Generalized Continuum Hypothesis Paradox, and The ∞ + 1 Paradox

$\aleph_n + 1 > \aleph_n$ for any cardinal number n, and $\infty + 1 > \infty$ for any ∞, even for Cantor's absolute ℵ/infinity.

❋ Remember: the reason these are all referred to as "paradoxes" is that they are such in any standard modern variant of Cantor's set theory, e.g. ZF and ZFC.

Historically, the Continuum Hypothesis question really depended on finding the least rapidly increasing successor function for transfinite numbers that yielded an obviously larger transfinite number (or cardinality). It was historical happenstance that lead to the power set *seeming* to be that function. But Cantor et al had a psychological and/or spiritual need to have an infinity that could not be made larger by adding 1. This is hard to understand, and it is even harder to understand why they didn't analyze that concept using at least something like the bijectivity preserving permutations of bijections presented above. It is now rather clear that it is the *assumption* that transfinite sets can be bijected with proper subsets of themselves (ostensibly proven in standard set theory, but in fact provably false as seen above; see also next section) that leads visibly to the Continuum Hypothesis, and invisibly to "problematic" paradox, i.e. inconsistency of the fatal sort since set theory is not paraconsistent, and probably cannot be made so successfully.

For completeness we should make formal:

Theorem 13: The Inconsistency of Set Theory
Set theory, which holds that there exist bijections between sets and proper subsets of themselves, is inconsistent.

Proof: there is an assumption in set theory, i.e. an implicit axiom, that passes as a theorem that there exist bijections between sets and proper subsets of themselves, an assumption/theorem that is provably false. That it is false follows from Theorems 10-12. (This implicit axiom is a hidden, false assumption in a theorem that uses a variant of circular reasoning to prove a recursively equivalent assumption as the theorem; see "Recursively Circular Reasoning", page 40.) Technically, since this contradiction does not derive from the *explicit* formal axioms and rules of inference of set theory, this does not make set theory standardly inconsistent, but rather the victim of a hidden false assumption (only implicitly and informally an axiom) in a botched proof. (See next section for details.) But... close enough as no matter.

- **We *must* remember that a mathematical theory is standardly inconsistent if it is *possible* to derive a contradiction, i.e. both a theorem and its negation.**

In this regard Theorems 9-13 take us beyond "mere paradox", even beyond standard theoretical inconsistency. They demonstrate the extreme desirability of a new, non-standardly non-Cantorian set theory.

The Bijection Permutation Paradox 1, 'Counting' and Cantorian 'Reordering'

The Bijection Permutation Paradox 1, which corresponds to The Golfer's Paradox (page 13) shows up in the $n \to n + 1$ mapping of set theory that is used in the construction of the "bijection" between e.g. $\mathbb{N} \cup \{0\}$ and $\mathbb{N}$. We need to examine this $n \to n + 1$ mapping to see if we can find any flaws in the reasoning corresponding to

The Good Shepherd's Paradoxes and to Theorems 8-13, above, and it is actually quite easy to do.

Although the $n \rightarrow n + 1$ mapping may seem to be validly "simultaneous" across the whole set ℕ of the natural numbers, if we follow the fundamental principle of being able to go back to/to reinsert the original definitions/constructions in our derivations, we find that fatal flaws stand out. To start this examination, we will look at the identity bijection from the already constructed ℕ onto itself, and the standard Cantorian "reordering" of it to make room for e.g. the 0 of ℕ ∪ {0}.

Cantor said that the reordering could be accomplished by remapping 1 (implicitly from the subbijection from {1} onto itself) to 2, remapping 2 (also implicitly from the subbijection from {2} onto itself) to 3, etc. (He didn't emphasize any special need to do this "simultaneously".) Subtle, at first, but then manifestly obvious is that, when we do this remapping/reordering, we have a "different situation" than when we first created ℕ.

When we first created the set ℕ, we would create an element to put into the set (usually by taking the "successor" of the most recent element put in the set), we would implicitly make a new place for the element in the set (like providing a new empty box/slot for it with the new element's name/number on it), and we would put the new element in its place in the set, pairing it in an identity subbijection with its own box, thus constructing the next stage of the initial "primordial" identity bijection of the completely constructed ℕ (an implicitly constructed bijection, but standardly "completed"). Each time (with every new element and place for it) we increase the set's cardinality.

But since the set ℕ is "now" already completely constructed, a "completed" infinity, we have an unclean slate (no empty places/boxes/slots, and adding a new place/empty box/slot will be Cheating, and worse, will increase the cardinality of the set and we fail immediately). We have e.g. 1 already bijectively mapped from itself onto itself. We must "de-complete" the set ℕ, then "de-biject" or "de-link" this 1 from itself, destroying the bijection from ℕ to itself, and without having created the bijection from ℕ onto ℕ – {1} that we need to *validly* complete the Cantorian reordering. We also have on our slate 2 already bijectively mapped from itself onto itself. We must either "temporarily" create a non-bijective 2-to-1 submapping from {1,2} onto {2}, or de-biject/de-link 2 from itself before re-bijecting 1 onto 2, thus "temporarily" creating a non-bijective 1-to-0 submapping from {2} onto { } (before creating the {0} onto {1} subbijection, the completion of the first stage of Cantor's "reordering"). Neither of these gives us a valid intermediate bijection, let alone the needed *valid* final bijection from ℕ onto ℕ – {1}.

The problem is that, even though for every n there exists an $n + 1$, each and every one of them is already "married", bijected/linked from itself onto itself in the initial "primordial" identity bijection that we are "reordering". By finite induction it is easy to show that we will always have either a non-bijective 1-to-0 submapping, i.e. an unbijected/unlinked number (like the 0 that we started with), or a non-bijective 2-to-1 submapping. (We can safely ignore all the other such possibilities at this time.) And neither of these allows the construction of a valid bijection from ℕ onto ℕ – {1} that we need to complete the Cantorian reordering for Cantor's completed infinity of natural numbers. This analysis is clearly a formalized variant of The Golfer's Paradox (page 13).

Set theory hasn't formally-theoretically defined a "simultaneous mapping" in terms of the original sequential definition/construction as is standardly required. If we go ahead and try to perform the $n \rightarrow n + 1$ mapping "simultaneously", we either fail outright as demonstrated by Theorem 11, or we have to go back to the initial construction of ℕ. This again gives us 2 alternatives:

1) We can de-complete ℕ, create a new place in it for a new element, and put there whatever element is currently homeless and being juggled, then re-complete ℕ. This doesn't work because it is Cheating; we fail outright.
2) We can de-complete ℕ, create a new place in ℕ with a new element in it, already "married to itself" in the now extended (no longer "primordial") identity bijection of ℕ, then re-complete ℕ. (This is much like Cantor's transfinite ordinal $\omega + 1$.) But its creation/construction does us no good because, not only is it Cheating, it doesn't give us an *empty* place to put the currently homeless element. We are again "borrowing from Peter to pay Paul".

Remember: Cantor wanted his transfinite sets to be "completed infinities", i.e. to not keep having elements added to them. We cannot legitimately "de-complete" ℕ, but we must if we try to take the "simultaneous" approach when we don't accept that the sequential approach is *validly* unsuccessful. The Good Shepherd's Paradoxes and Bijection Permutation 2 (in particular, Theorems 8-12, pages 30-35) clearly demonstrate that we cannot successfully proceed in this fashion. In sum, and repeating for emphasis:

❃ There is a serious problem here: when we try to use the $n \rightarrow n + 1$ mapping to construct a bijection from ℕ onto ℕ – {1}, we either try to do it in the theoretically unimpeachable sequential fashion, or we try to do it using the "extra-theoretical" "simultaneous" approach. In either case, either we try to add a new element to ℕ, or we don't. If we don't try to add a new element, we fail immediately, as formally demonstrated by Theorems 8-12, as well as informally by The Golfer's and The Banker's Paradoxes. If we do (try to) add a new element, either we add it by constructing a new empty place for it but "Vanishing" the new element and putting in its place whatever element (perhaps the 0) is currently homeless in our ongoing attempt at "reordering" (outright Cheating; we fail), or we add a completely new element in the empty place and we are in the same boat as when we hadn't added a new element (as seen in The Golfer's Paradox, page 13,); we fail yet again. Further, if we make *any* attempt to add a new (place for a new) element in any sense, the already completely constructed, finished, formally "completed" set ℕ must be "de-completed", violating its earlier Cantorian "completeness", and when we "add the new place for an element, empty or not" and then "re-complete" it, we continue to violate its earlier Cantorian "completeness". We make ℕ both "non-self-equal" and "non-self-equivalent" as we engage in these Cantorian "reorderings" that pervade set theory. And unlike the summing of 2 infinite sequences, e.g. $a_i + b_i$ for all i in ℕ, that can be "completed" or "finished" in $\aleph_0$ operations, the $n \rightarrow n + 1$ mapping cannot "completely" construct a bijection from (strictly) ℕ onto (strictly) ℕ – {1} in $\aleph_0$ operations, or in *any* number of operations. (The Golfer's Paradox, page 13, gives a much more intuitive demonstration of all this.)

- ❈ Whether we misguidedly try to add a new element to ℕ or not, we have here a standard "borrow from Peter_{n+1} to pay Paul_n" situation (even if we also need to eventually "borrow from Paul to pay Peter"). This is in fact the simplest variant of that common swindle, a "Pyramid Scheme", although not the usual exponentially increasing kind. It is perhaps the least ambitious such fraud, unless we construct a Hilbert Hotel based on this process. "But t'is enough; t'will do..."

These insights are essential for the development of new paradigms for a new, non-standardly non-Cantorian set theory.

Theorem 14: The Bijection Permutation Paradox 1: The $n \to n + 1$ mapping.
The standardly accepted $n \to n + 1$ mapping cannot construct a bijection from ℕ onto ℕ – {1}, or from ℕ ∪ {0} onto ℕ, in $\aleph_0$ operations, or in any number of operations. Stated without further proof. (See The Golfer's Paradox, page 13.)

It also helps to look at the squares paradox associated with Galileo. Galileo noted, as the ancients probably did also, that there seem to be "as many squares of counting numbers as counting numbers", i.e. that it seems to be possible to put the counting/natural numbers into a 1-to-1 correspondence with their squares, 1↔1, 2↔4, 3↔9, etc. (We could also use the seemingly equal numbers of even numbers and counting numbers, 1↔2, 2↔4, 3↔6, etc, but squares make the arguments somewhat clearer.)

We need to go back to the construction of {1,2,3...} again. When we have constructed 2, we have not yet constructed the square of 2, $2^2 = 4$ (nor have we even constructed the successor of 2, 2 + 1 = 3). As with the mapping $n \to n + 1$, we need to borrow from the "future", now all the way to n^2 instead of merely to $n + 1$. (If we were working with power sets, we would need to borrow all the way to 2^n.)

Since we are comparing the counts of the numbers in this case instead of trying to make room for only one extra number, 0, we need to look at the situation a little differently. By the time we have constructed e.g. $5^2 = 25$, we have constructed 25 natural numbers but only 5 squares of natural numbers. (Alternatively, when we have constructed 5 natural numbers, to get the squares we have Borrowed-Embezzled-Juggled into the future all the way to 25, for a Net Profit of 20.) This generalizes so that when we have constructed n, we have approximately (~) $\sqrt{n}$ squares. The relative number of squares increases as $\sim\sqrt{n}\,/n$, as should be obvious.

- ❈ When we analyze in this more careful "Auditing" fashion (suggested by The Banker's Paradox), the fallacy behind Galileo's paradox stands out. The resolution of this paradox (and many others), and similarly of the paradox associated with the $n \to n + 1$ mapping, is that we need to stay *within* the bounds of the already constructed n as we let it increase to "infinity". (Gödel's work, for example, will need to be reinterpreted in this light.)

This same analysis applies to Cantor's diagonalization technique that has become paradigmatic throughout mathematics (e.g. in automata theory, where, Paradoxically, it is used to prove the existence, not of an uncountably infinite number of computable functions, but the existence of "*non*-computable" functions). Cantor takes the first real number (decimal places to the right of the decimal point only) and starts making a number that differs from that first listed number in the first decimal place. But if we are dealing with decimal reals, the entire list of one decimal place reals would be 10 reals, and all that Cantor has really done is to produce a real that has not occurred in the first place in the list; it certainly occurs later in the *complete* list of these 10 reals. Now we consider two such decimal places: Cantor produces a real that has not occurred in the first two places in the list, but the list is now of 100 reals, and again we find (that much of) it later in the list. The error that Cantor made was to think that the final length of the list must be $\lim_{n\to\infty} n = \lim_{n\to\infty} 10^n = \infty = \aleph_0$ instead of $\lim_{n\to\infty} 10^n = 10^\infty = 10^{\aleph_0} >>> \aleph_0$. (For Cantor, the $\lim_{n\to\infty} 10^n = \aleph_0$, not anything like the uncountable infinity of the reals, which for him was $10^{\aleph_0} = 2^{\aleph_0}$, which uncountable infinity can apparently only be constructed "simultaneously". So even though the list was of reals with $\aleph_0$ decimal places of 10 possible values each, the list remained a maximum of countably infinite, i.e. $\aleph_0$, listed real numbers in length.)

It is yet another case of attention to detail. This analysis is paradigmatic for a new non-Cantorian set theory, and so worth the extra attention at this time.

We can look ahead a bit more to new paradigms and paradigm attributes for counting in a new theory. We will have that $n + 1 > n$ for all n, "finite" or "transfinite" (though just what those will turn out to be is still vague). We therefore won't have to worry about the $1 = 0$ paradox that we currently have, that brings us e.g. the Banach-Tarski paradox of the now standardly accepted paradoxical measure theory, and the need for renormalization in physics (e.g. in quantum mechanics).

This lets us note something obvious about using a set to count (using the usual bijective 1-to-1 mappings):

❋ If we have a set of cardinality n (using standard terminology for now), we can count "carefully" or "precisely" only up to and including n - 1. But if we have counted all the way to n, we can now only have a "lower bound count" since we have no way of knowing if there were yet more elements that went "uncounted". We need to either be careful to count "carefully", e.g. not to Borrow-Embezzle against the future of n greater than (or even equal to) our current counting set maximum, or at least to be careful to notice when we exceed our counting set, as in the Galileo Paradox where we outstrip infinity by quite a bit. We can now begin to speak of the "accountably infinite", where not even 1 gets lost.

Continuum Hypothesis Paradox 4: The First Uncountable Infinity: $\aleph_0$ + 1.

Because $\aleph_0 +1 > \aleph_0$, $\aleph_0 + 1$ is the first literally uncountable infinity.

This all deserves much more serious attention than can be given here.

"Recursively Circular Reasoning"

From a philosophy of mathematics point of view, we can note that there is circular reasoning involved, but of a somewhat subtle recursive nature. In the standard proof that there exists a bijection from $\mathbb{N} \cup \{0\}$ onto $\mathbb{N}$, we effectively recursively assume that there exists a bijection from $\mathbb{N}$ onto $\mathbb{N} - \{1\}$, the use of which assumption (which passes as a "simultaneous" mapping from $n \to n + 1$, the extra-theoretical nature of which is still Overlooked) makes that proof seem as trivial as constructing a bijective mapping from 0 onto 1. (I.e. once we have the "bijection" from $\mathbb{N}$ onto $\mathbb{N} - \{1\}$, then the "divorced" 1 is obviously "available" for "remarriage" to *any* element not in $\mathbb{N}$.) We are never called on to do so, but if we did have to prove that there exists a bijection from $\mathbb{N}$ onto $\mathbb{N} - \{1\}$, we would have to recursively assume that there exists a bijection from $\mathbb{N} - \{1\}$ onto $\mathbb{N} - \{1,2\}$, and so on. At least one of this recursively defined set of assumptions (in counterpoint to a transfinite schema) must be implicitly postulated, making it in fact a hidden axiom, and, most unfortunately, a provably false one since:

1) The "simultaneous" mapping from $n \rightarrow n + 1$ fails when analyzed by strict mathematical requirements, such as the necessity of referring to the initial definitions/constructions, and always being able replace defined entities by these same original initial constructions/definitions. (See "The Bijection Permutation Paradox 1, 'Counting' and Cantorian 'Reordering'", page 36.)
2) **So there *never was* a valid proof that $\aleph_0 +1 = \aleph_0$.**
3) Theorems 6-12 (pages 29-35) are quite cogently provable, formally and rigorously.
4) **So there *is* a valid proof that $\aleph_0 +1 > \aleph_0$.**
5) **Many of the Overlooked Paradoxes derive from this hidden, false axiom (that sets can be bijected onto proper subsets), and from the "Recursively Circular Reasoning" needed to "prove" it as a theorem.**

It is worth mentioning at this point that we have the beginnings of:

- **A Resolution of The Bijection Permutation Paradoxes**
  We have the beginnings of a viable, if almost certainly sweeping, resolution of The Bijection Permutation Paradoxes. If concurrently we can properly resolve The Vanishing Remainders Paradoxes, which only partly derive from them, set theory has a chance to be made consistent. To make good on that chance, problems with the Axiom of Infinity need to be concurrently resolved, as well. (See the "The Maximal Element of $\mathbb{N}$ Paradox, and The $\aleph_0$ As A Natural Number Paradox", page 47.)
- **Reminder: if one has already proven a theorem true, this does not mean that a later proof of its negation (or of its falsity, if different) must be invalid. It firstly means that both proofs must be carefully reexamined. Secondly, if they both validly derive from the initial formal axioms and rules of inference of the theory, then the theory must be found, by definition, to be inconsistent.**

But, blinded by the light of Cantor's paradise, we have heretofore failed to go back to examine certain standard fundamental proofs in light of the initial serial constructions/definitions, a course that helps make these failures, such as the use of Recursively Circular Reasoning, stand out when we carefully reexamine the foundations of set theory. We wanted infinity, and to us that meant that infinity should not, could not have a "maximal element". That was in ancient times when we were thinking of infinity as forever "unfinished", i.e. forever "incomplete". And we constructed a mechanism that always produced (a set with) a maximal element. But simultaneously, we (well, Cantor et al) wanted a "completed" infinity, *without* a maximum element. These desires seem to be irreconcilable.

More Real Number Theory Paradoxes

There are still many Overlooked Paradoxes that are worth exploring. We will now go back to others in the Vanishing Reminders family of paradoxes, or closely related.

Standard theory holds that between any 2 rational numbers there exists a real number. But it also holds that between any 2 reals there exists a rational. The "Non-Denumerable Rationals Paradox" has been overlooked.

Since there are non-denumerable *distinct* reals in e.g. the unit interval, and each of these reals is either greater than or less than all the others, there must also exist a non-denumerable partition (very many such, actually) of the unit interval, each subinterval of which has non-rational "endpoints" and contains non-denumerable reals (although neither of these points is strictly essential to the argument), and, Paradoxically, at least 1 rational number. I.e. by standard theory, even though Overlooked, it is provable that there must be non-denumerable rationals, in counterpoint to provably having countable reals (see page 9).

Non-Denumerable Rationals Paradox

Since between every two real numbers there standardly exists a rational number, the rational numbers must *also* be non-denumerable (as seen above).

For those who like to trace the origins of paradoxes, it turns out that this last paradox derives pretty much directly from the Archimedean property of the real numbers. (See page 43 for more details.) If one has a transfinite number of distinct numbers (or points) in a finite interval, at least some of the distances among them must be non-Archimedean, and at the same time, must be real numbers (in the field of the reals, $\mathbb{R}$).

Although the argument is informal for reasons of space, the "Vanishing Remainders Paradoxes" (see page 2) and "Countable Reals" Paradoxes (see page 9) combine to give us the "Cauchy-Dedekind paradox". I.e. they tell us that Cauchy sequences of rationals and Dedekind cuts (using 2 sequences of rationals, one less than and one greater than the irrational number being "uniquely" defined) cannot successfully define all real numbers. This is because a sequence of (standard) rational numbers has 1 as its smallest (positive) non-zero numerator, and n "as $n \to \infty$" as its largest denominator/divisor, necessarily combined with the smallest non-zero "Vanishing Remainder" of 1 (which shares that divisor), giving us the "smallest possible (absolute value) distance between unequal rational numbers". (This is only for standard rationals with the same denominator. If we take into account different denominators, we could get the limit of $n \cdot (n - 1)$, or even $n!$, "as $n \to \infty$" as its largest denominator/divisor. Standard theory holds that we must still get $\aleph_0$ as that limit, but we should have begun to question that.)

We can informally say that this smallest non-zero distance, we can call it the "granularity" of the rationals, is (approximately, for many reasons) "$1/\aleph_0$", as when we constructed a denumerable closed cover for the unit interval. Unless we really want non-denumerable rationals, then we *must* have rational granularity intervals containing $\sim 2^{\aleph_0}$ reals but not containing any rational numbers.

- ❈ A "standard rational" can have a numerator or denominator no greater than $\aleph_0$, and will have a "granularity" of "$1/\aleph_0$". If we allow the limit of e.g. $n \cdot (n - 1)$ or $n!$ "as $n \to \infty$" (i.e. if we allow the cardinality of the rationals) to become greater than $\aleph_0$, we exit standard set/real number theory but get other possibilities, e.g. conceivably a *much much* smaller "granularity".

Not even an infinite sequence of *standard* rationals can get us unambiguously and indefinitely closer to precisely one *arbitrarily chosen* standard real number in that interval of $\sim 2^{\aleph_0}$ reals than that "granularity" suggests. E.g., between 0 and "$1/\aleph_0$" there must be $\sim 2^{\aleph_0}$ "quantinuous" reals with *no* rationals, but a sequence of rationals can only "converge" to either the 0 or the "$1/\aleph_0$", nowhere in between (except e.g. by interpolation). (See "The Quantinuum Hypothesis", page 45.)

"Information theoretically" speaking, infinite sequences of rationals should allow us to *encode* $2^{\aleph_0}$ real numbers (with something akin to the topological discontinuities we get when projecting spaces onto lower dimensional subspaces), but the arithmetic convergence requirement takes away that ability. The reals (must) have their own granularity ("$1/2^{\aleph_0}$", or "$1/10^{\aleph_0}$", etc), and one of the casualties here will be the Bolzano-Weierstrass theorem with its "cluster points", since the granularity that derives from the actually non-vanishing remainder ensures that we can always find a non-null punctured neighborhood around any point with no numbers/points in it whatsoever. Let's amend that somewhat: a real number granularity neighborhood of width "$1/2^{\aleph_0}$" *could* have e.g. $\sim 2^{2^{\aleph_0}}$ "giga-reals" in it, etc, but only *if* the system of numbers is so constructed. This all depends on Cantor's "completed infinities", but they exist in any standard set theory.

(It is also possible that someone will try to hold, contrary to standard theory, that the reals are really just extended rationals with infinite numerators and denominators, and thus such sequences *can* converge to precisely 1 real number. This is closer, but... still no cigar. Such a position destroys standard set theory, and still does not take into account Vanishing Remainders and Real Number Paradoxes 1-3.)

Cauchy-Dedekind Reals Paradox

The real numbers cannot be completely defined by Cauchy sequences of rational numbers nor by Dedekind cuts.

This set of issues also relates to the **Archimedean property**, a standard property of the real numbers (and thus of all finite cardinal numbers, etc.) By the standard Archimedean property, "for any positive reals *a* and *b* there exists an *n* such that $n \cdot a > b$." (All 3 numbers are standardly assumed to be "finite", if only implicitly.) It is well-known that the Archimedean property standardly does not allow the existence of infinitesimals (since *n* must be "finite"), and Paradoxically Overlooked that it does not allow the existence of *standard* set theory's infinite/transfinite cardinals (or ordinals), either. Actually, it ensures that one gets neither or *both*, but the allowable transfinites would need to be non-standard. (See Diagram 2, page 10, for a possible paradigm for making transfinites and "transfinitesimals" Archimedean.)

- **The Vanishing Infinitesimals Paradox 3: The Archimedean property, as it is standardly utilized in conjunction with $\aleph_0 + 1 = \aleph_0$, which allows $n\aleph_0 = \aleph_0$ and effectively makes $\aleph_0$ non-Archimedean in the same way that the otherwise standardly real number 0 is non-Archimedean, effectively prevents the field of the reals (ℝ) from being extended to "transfinitesimals" and "transfinites" (which many have thought they would enjoy). I.e. when the 1 of $\aleph_0 + 1 = \aleph_0$ Vanishes on the other side of the equation, the possibility of any theoretical development of infinitesimals, or "transfinitesimals", and transfinites as real numbers Vanishes with it.**

And we should not overlook:

The Archimedean Limits at Infinity Paradox

The Archimedean property gives rise to Paradox with regard to real number theory's ubiquitous "limit(s) as *n* goes to infinity". All real numbers (except 0, technically) standardly have the Archimedean property. Not only does the Archimedean property make it theoretically impossible for infinitesimals to exist, it has been Paradoxically Overlooked that the Archimedean property also makes it theoretically impossible for currently standard transfinite numbers to exist. Thus the *real* variable *n* takes on a transfinite and "un-real" value when taking a limit "as *n* goes to (a standardly "completed")

infinity". The mere existence of a transfinite number of numbers/points in a finite interval means that there must exist (non-zero but) non-Archimedean distances among at least some of them (in neighborhoods around cluster points). Formally, this existence is strictly "extra-theoretical", i.e. strictly "iffy-theoretical": we must expect to find "Paradoxical" consequences, and do...

One Paradoxical consequence is that one can also derive the "Countable Reals Paradox 1" directly from the Archimedean property:

Theorem 15: The Archimedean Property of the Reals Paradox

If the real numbers (ℝ) are standardly non-denumerable, then they cannot have the standard Archimedean property. If the real numbers are standardly Archimedean, then they cannot be standardly non-denumerable, but rather must be standardly countable.

Proof: we can rewrite $n \cdot a > b$ as $n > b / a$, and note that n is an upper bound on the number of points between 0 and b that are separated by a minimum distance a; since it retains this property for *all* $a > 0$, yet again the reals, which standardly have the Archimedean property, are paradoxically also provably countable.

The existence of $2^{\aleph_0}$ reals evenly distributed in e.g. the unit interval should have given us the certainty that there must be non-zero differences among them as small as 1 (> 0 since we don't want the Remainder to Vanish) part in $2^{\aleph_0}$, a distinctly non-Archimedean value.

We can see that the Archimedean property of the reals helps give rise to the Non-Denumerable Rationals Paradox. The Archimedean property does not allow standard "completed infinities", although it does allow numbers to get "larger and larger" (and, correspondingly, "smaller and smaller"). Since it doesn't allow (standard) denumerable infinities, it certainly doesn't allow the (standard) non-denumerable infinity of the reals. Thus the Archimedean property kind of assumes that any reals will be mixed in among the rationals with no bias as to frequency. (In between any two rationals is a real, and vice-versa, a very strange concept when we compare $\aleph_0$ with $2^{\aleph_0}$.) When the property of being non-denumerable is posited for the reals, out pops the non-denumerablity of the rationals. It doesn't matter that (standard) non-denumerable infinities are inconsistent with the Archimedean property; logic grinds away on the assumptions (axioms and other assumptions, even if hidden/denied) producing valid theorems, even if the theorems contradict each other and the theory is itself inconsistent, "technically".

- **Remember: a mathematical theory is "inconsistent" if it is *possible* to derive two *valid* theorems that contradict each other. The contradiction doesn't make *either* theorem or its derivation/proof "invalid". There is definitely a *psychological* primogeniture, but that is strictly "extra/iffy-theoretical".**

If we take into account the Real Numbers Paradoxes 1-3 (see pages 5-6), the Cauchy-Dedekind Reals Paradox (page 43) and Theorem 15: The Archimedean Property of the Reals Paradox (as seen just above), we then have the:

Real Numbers Paradox 4: Real Numbers Existence Paradox

The real numbers do not exist by any of the standard definitions in our current real number theory based on any standard (Cantorian) set theory.

Mild digression: $\aleph_0$ is not considered to be an integer in the field of the reals, even though it is considered to be the cardinality of the integers greater than 0 (etc.), and even though it is the ultimate successor (by adding 1) cardinal of 1, the "limit of *n* (or is it *n* + 1?!) as *n* 'approaches' infinity". It is worth noting that a field has much the same definition as the set of all natural numbers, since if *a* and *b* are field elements then *a* "+" *b* is also a field element. If by definition a field allows "any number" of compositions of elements, then it follows that it allows any number of "+ 1" equivalents and therefore allows a field theoretical equivalent of $\aleph_0$ to be an element. Mathematicians try to back away from such reasoning by insisting that only "all finite" combinations/successions are being considered, but the definition/construction of the natural numbers and finite induction also are brought to us by and give us (respectively) "all finite" combinations/successions. Since the standard construction of the natural numbers and the co-construction of finite induction give us $\mathbb{N}$, they should also give us $\aleph_0$, but by standard theory they don't. If pressed, mathematicians will probably try to insist this is mere "paradox", not "inconsistency". End of digression.

Real number theory has some serious paradoxes in need of public recognition (as do measure theory, analysis, et al). We need to accent that both real number theory and these paradoxes derive from Cantorian set theory, and relate to the results in Theorems 9-12 (relating to The Bijection Permutation Paradoxes; see pages 33 and 36). It becomes obvious why we should start to develop non-Cantorian variants of set theory, real number theory, measure and integration theory, analysis, topology (some), etc. But, we need to find the "source of the Nile", in our case the "source of the Vanishing Remainders Paradoxes".

The Quantinuum Hypothesis

A quick look into the future: the "Quantinuum Hypothesis" as an alternative to the Continuum Hypothesis is here seen to be inevitable since the Continuum Hypothesis really depends on the Remainders Vanishing (and staying Vanished). In a new, non-standardly non-Cantorian set theory we will need to develop the concepts of "quantinuum", or rather of many potentially "incommensurate" "quantinua" (see the Real Numbers Paradox 3, page 6), along with "quantinuities", "disquantinuities", etc. (A non-standardly "non-Cantorian set theory" will be needed since non-Cantorian set theory is currently defined purely by the axiomatic rejection of the Continuum Hypothesis, not its falsification within the theory.) The concept of "the continuum" as we know it will be a casualty, since once we speak of "the continuum" as a "number" of points/numbers, even an "absolutely infinite" number of points/numbers, we must perforce have a "quantinuum". "Open" and "closed" sets, "dense-ness", and many other concepts will not be continuing on, either, as will become obvious.

As we analyzed in detail, above, even Cantor's "absolute infinity" can be made greater by adding 1, and even absolute infinities used as a divisor leave that strictly Non-Vanishing Remainder of 1 that gives us a non-null punctured neighborhood around any point that does not have any numbers/points in it, characteristic of a "quantinuum". Any "granularity" leaves transfinitely many "vast empty spaces" of transfinitesimal size. A true continuum could have a transfinite number structure placed on top of it that would still leave these transfinitely "vast empty spaces" capable of holding transfinitely many, transfinitely vaster such structures. (We could say that placing the real number structure on top of the structure of the rationals is a quasi-example of this.) We mistook our actually "quantinuous" such structure(s) for "the continuum".

Continuum Hypothesis Paradox 5: "The Continuum-Quantinuum" Existence Paradox:

A "true continuum" would require a "granularity" (as informally defined above) of absolute zero, i.e. no granularity whatsoever, with no "Vanishing Remainders" and the "quantinua" that perforce come with them. Even "1 / Cantor's absolute infinity" has a strictly non-zero granularity, so in fact "The Continuum" as currently conceived in set theory can not truly be a continuum, nor can any "Generalized Continuum" (page 36). We must have a "Quantinuum", or rather many possibly incommensurate "Quantinua".

And Non-Vanishing Remainders perforce give us not only quantinua, but naturally theoretically derivable infinitesimals, or to put Cantor's terminology to new use, "transfinitesimals", of many orders if we retain anything like Cantor's hierarchy of transfinite cardinals. As with quantum mechanics, we will need to start distinguishing properties (the "classical" properties that we are more familiar with) from the micro-properties, e.g. the "local quantum micro-topologies", and from the macro-properties and macro-topologies of the transfinite realms. (And we also need to start distinguishing such macro-properties in physics, e.g. in cosmology. If quantum mechanics is so different from classical mechanics, there is no reason why we should not look for "macrocosm mechanics" that is "equally different". We could avoid futile, time wasting attempts to study macrocosmic scales purely in terms of classical physics.)

Other Set Theory Paradoxes

Even though standard set theory is not based on a paraconsistent logic, it makes sense to look for other set theory "paradoxes" that might derive from, or otherwise relate to, the main results above. These next paradoxes have probably been noticed by many (e.g. undergraduates who didn't yet know "better"), and many times, but they have never been publicly analyzed and satisfactorily resolved. They are the kind that is easy for *trained* mathematicians to dismiss out of hand. They are also much easier to accept once the paradoxes just analyzed in detail, above, are accepted.

One paradox that The Bijection Permutation Paradoxes help explain is the:

Transfinite Cardinal-Ordinal + 1 Paradox

When we add elements to a set under construction 1 element at a time, do we add them cardinally or ordinally? or do we get a choice? We get very different results, depending. E.g. cardinally, one can add 1 (new and unique element) *any* number of times to (a set of cardinality) $\aleph_0$ (or any transfinite *cardinal*) without standardly increasing the cardinality, since it is always the case that $\aleph_0 + 1 = \aleph_0$ *every* time we add 1. But, paradoxically, in Cantor's (and e.g. ZFC's) ordinal arithmetic, if we add 1 enough times to the first transfinite ordinal ω, which has cardinality $\aleph_0$, we eventually wind up with the transfinite ordinal Ω, which has cardinality $2^{\aleph_0}$, and so on to Cantor's absolute infinity. (The Paradoxical Question comes up of what happens if we perform Cantorian reorderings of the ordinal set before it gets to Ω. It's still the same set...) This extends to all corresponding transfinite cardinals and ordinals.

A minor variant involves showing that the same set of elements can be both non-denumerable and denumerable: if we apply a well-ordering to any non-denumerable set, we can take the elements out of it one at a time (by

choosing the current least element) until the set becomes empty. If we (cardinally) add each element as it is taken from the non-denumerable set to a set that is initially empty, then since we are adding the elements one at a time, the cardinality of the new set can only reach $\aleph_0$ as a maximum. (If we are "adding the elements ordinally", the set can even be Cantorianly "reordered", as Cantor did with his transfinite ordinals, before any problems start to emerge.) But of course the new and strictly denumerable set has precisely the same elements as the old non-denumerable set.

One paradox that The Bijection Permutation Paradoxes help explain is the:

Transfinite Cardinality Element Subtraction Paradox

If we have the set of natural numbers, $\mathbb{N}$, and we subtract the element 1 from it, the cardinality of the resulting set is still $\aleph_0$. If we have subtracted the first *n* elements from $\mathbb{N}$ and the resulting cardinality is $\aleph_0$, then if we subtract the element $n + 1$ the resulting cardinality will remain $\aleph_0$. So by standard finite induction we can subtract all the elements from $\mathbb{N}$ and the cardinality of the resulting set, the empty set, must still be... $\aleph_0$.

Two related paradoxes that seem to derive from the Axiom of Infinity, and seem to make set theory legitimately inconsistent, are the:

The Maximal Element of $\mathbb{N}$ Paradox, and The $\aleph_0$ As A Natural Number Paradox

$\mathbb{N}$, by definition, cannot have a maximal element. The paradox is that $\mathbb{N}$ is also, by definition, the ultimate successor set to {1}, and, by finite induction (and by "Theorem 1a: "Transfinite Case" Finite Induction; see page 23), the set {1} and *every* successor set of {1} must have a maximal element, which maximal element is both the cardinality of that same set *and* a natural number; this includes the "ultimate" successor set $\mathbb{N}$ and its cardinality $\aleph_0$.

Instead of referring to this as contradiction and inconsistency, with the finger pointing squarely at the Axiom of Infinity, we can refer to it as an... Unappreciated, if not Overlooked, Paradox.

On the other hand, we need a new Axiom of Infinity. The importance of this flaw – and it is actually fatal – in our system of axioms and rules of inference cannot be overstated.

Some think $\aleph_0$ should be accepted as a natural number, but to do so would disrupt set theory so much that it would effectively necessitate a new and non-Cantorian set theory. In current set theory it is legitimate to define entities that do not exist. It is even legitimate to define entities whose existence would mean that set theory was inconsistent, requiring one to refrain from axiomatizing, postulating or otherwise positing their existence. If one then somehow does posit the existence of such an entity, the theory immediately becomes inconsistent. The Axiom of Infinity, unfortunately, clearly seems to posit the existence of such an entity. It would also be possible to forgo $\mathbb{N}$'s being considered a "successor set" of {1}, for example by not considering it to be a "set", but that too would disrupt set theory rather completely.

And a final paradox that has perplexed many beginning students of set theory. It concerns a repercussion of $\aleph_0 + 1 = \aleph_0$, and is perhaps the most amusing Paradox of Infinity to date:

The Transfinite Cardinal-Ordinal Exponentiation Paradox

Set theory's standard transfinite arithmetic holds, for transfinite cardinals, that $\aleph_0 + 1 = \aleph_0$, and that it leads to $\aleph_0 + n = \aleph_0$ which leads to $2\aleph_0 = \aleph_0$, $n\aleph_0 = \aleph_0$, $\aleph_0^{\,2} = \aleph_0$, $\aleph_0^{\,n} = \aleph_0$, $\aleph_0^{\,\aleph_0} = \aleph_0$ and so on. But since $\aleph_0^{\,\aleph_0} = \aleph_0$, and $\aleph_0$ is $<<< 2^{\aleph_0}$, we get the amusing Paradox that $\aleph_0^{\,\aleph_0} <<< 2^{\aleph_0}$. Similarly, the transfinite ordinals (which are sets) such as ω^{ω} $\omega^{\omega^{\omega^{\cdots}}}$ are all of cardinality $\aleph_0$, until one gets to Ω (the first transfinite ordinal that has cardinality $2^{\aleph_0}$). If one wants to standardly exponentiate 2 sets, *a* and *b*, as a^b, the fundamental set exponentiation theorem gives the cardinality of a^b as: $|a^b| = |a|^{|b|}$. From this we can standardly get $|\omega^{\omega}| = |\omega|^{|\omega|} = \aleph_0^{\,\aleph_0}$, but since we already have that $|\omega^{\omega}| = \aleph_0$, we immediately, by a second route, get that $\aleph_0^{\,\aleph_0} = \aleph_0$ and therefore $\aleph_0^{\,\aleph_0} <<< 2^{\aleph_0}$.

It's time to remind ourselves yet again of a fundamental mathematical reality:

- **ESSENTIAL: If we always formally invalidate the proof or derivation of a new theorem merely because the new theorem contradicts an already proven theorem, and thus would show that our theory is inconsistent... then ANY theory is "provably" "consistent" (Gödel to the contrary notwithstanding) by the rule of "psychological primogeniture".**

Quick Remarks on the Axiom of Choice

Zermelo (Ernst Friedrich Ferdinand, 1871-1953), the first to develop an axiomatic set theory (1908), did so in a restrictive way so as to try to make impossible the derivation of the then known contradictions that were found in Cantor's more informal theory. (Later improvements suggested principally by Skolem and Fraenkel were eventually included, but for some reason only Fraenkel's name is included with Zermelo's. The C in ZFC means that the Axiom of Choice is included with the ZF axioms and rules of inference.) Zermelo was concerned with the construction of problematic sets such as set of all sets of Cantor's paradox, thus the ZF axioms are restrictive regarding such constructions. To get around these rather severe restrictions, mathematicians later added the Axiom of Choice, allowing constructions that to some seemed to open the door again to contradiction... and possible inconsistency.

If Cantor's set theory was inconsistent before ZF (and this clearly seems to be the case), and the basis for this inconsistency was incorporated unknowingly in ZF, but the contradictions needed the constructions of sets not available under ZF to make them clearly apparent, then this would "explain" the reaction of some mathematicians that the problems that became apparent when the Axiom of Choice was included were/are the result of the axiom itself, and not of all the axioms and rules of inference taken together (and especially not of the earlier theory itself). This is a variant of "Blaming the Messenger".

A satirical analogy from the world of Banking suggests itself: if our "Books" don't balance but we don't know it, and we decide to posit an "Axiom of Auditing Allowed" that lets us examine the Books in detail (perhaps entailing the construction of Auditing Books that normally wouldn't be allowed), then we will note that "the new axiom causes all the problems", e.g. "Auditing causes Embezzlement."

This distinctly seems to be the case with the Axiom of Choice. There were preexisting flaws that started leaving trails of noticeable clues when we could construct the "Auditing Books" needed to display it clearly, but instead of finding fault with our

Pre-Auditing System, we blame the Auditing and/or Auditing Procedures for the Embezzlement.

If we have a theory of transfinite sets, then we need to be able to construct further transfinite sets to look more closely at the transfinite sets already accepted under that theory, to "Audit The Books", as it were. In a new and non-standardly non-Cantorian set theory, the Axiom of Choice question will definitely be a null issue: the axiom was never really needed, but merely a sign and/or function of our psychology in dealing with earlier but seriously overlooked theoretical failings.

The Desirability of a New, Non-Standardly Non-Cantorian Set Theory

We started with the Vanishing Remainders Paradoxes, easily accessible to all but still compelling, moved on to The Good Shepherd's Paradoxes, not only easily accessible but paradigmatically crucial, then to the Bijection Permutation Paradoxes, just about as simple and formally irrefutable as mathematics ever gets. Set theory, real number theory, and any other theory that depends on set theory's concept of infinity not being *too* paradoxical all find themselves in rather dire straits.

The Continuum Hypothesis has a resolution unforeseen by Cantor et al, in particular one that does not involve the axiomatic rejection of the Continuum Hypothesis, but rather a derived rejection of it, along with a new "Quantinuum Hypothesis" derived from newly discovered and otherwise irresolvable paradoxes. The resolution found more than suggests that some of the foundations of Cantorian set theory and its axiomatizations, whether ZF, ZFC, or BNG, must be fundamentally reworked, essentially from scratch.

The new (because Overlooked) Paradoxes presented in this paper need to be resolved. It will not suffice to avoid applying rules of inference, e.g. about subtracting equal quantities from both sides of an equation. Nor will it suffice to avoid replacing defined entities by their original constructions and then carefully reexamining the consequences. Again, it will not suffice to ignore the current definition of an "inconsistent" mathematical theory. And yet again, it will not suffice to blame the Axiom of Choice.

We have seen what can be considered the beginning of a viable resolution of The Bijection Permutation Paradoxes. If concurrently we can properly resolve The Vanishing Remainders Paradoxes, which only partly derive from them, set theory has a *chance* to be consistent. To make good on that chance, problems with the Axiom of Infinity need to be concurrently resolved, as well. (See the "The Maximal Element of $\mathbb{N}$ Paradox, and The $\aleph_0$ As A Natural Number Paradox", page 47.)

The chance to get in on the ground floor of developing a new set theory, one that must perforce differ essentially and drastically from the current Cantorian standard, will appeal to many, after the 5 stages of acceptance (denial, etc) work themselves out. Exploring a new resolution of the Continuum Hypothesis and the consequences of that resolution will be a compelling motivation for some, less compelling for others. An obvious, practical incentive for new set and real number theories is the possibility of successfully resolving the renormalization problem in physics, most notoriously in quantum mechanics. Developing a set theory without the misguided Axiom of Choice will entice some. But avoiding paradox that is rather too paradoxical will probably be the main driving force behind the future explorations that can now be seen to be inevitable.

It will be helpful to evaluate the possibilities of "paraconsistency" in a refurbished set theory (i.e. one or more of its major variants: ZF, ZFC, Bernays-von Neumann-

Gödel, etc.), and the two thirds of modern mathematics that have any variant of Cantorian set theory as an essential part of their foundations (along with logic, where paraconsistency is again a vital issue).

But it will help even more to start to explore the many currently non-standard possibilities for non-Cantorian set theories that are suggested by the paradigmatic results of Theorems 2-14, and their consequences in real number theory, analysis, measure theory, integration theory, various algebras and topologies, model theory, possibly category theory, and others.

New Theoretical Bases For a New Set Theory and New Theories Based On It

For example, in a non-Cantorian world we will probably want to look forward to:

- Set Theory: $|S \cup \{x\}| > |S|$ for any set S and any non-null $x \notin S$
- Number Theory: $n + 1 > n$ for arbitrary finite or transfinite n;
  perhaps something like $(n + 1) - n \equiv 1$ to try to ensure Euclidean linearity
- Measure Theory: $(\text{measure}(S) = 0) \equiv (S = \phi)$ (where $\phi \equiv \{\ \}$, the empty set)
- Measure Theory: for arbitrary finite or transfinite n, $\sum_{1}^{n} 1/n \equiv 1$ (Euclidean linearity, again, with non-integer or even negative n offering interesting possibilities)
- Number Theory: correspondingly, for arbitrary finite or transfinite n, $n \times 1/n \equiv 1$ (Euclidean linearity. NOTE that none of this emphasis on Euclidean linearity is intended to imply that we should overlook non-Euclidean, non-linear systems of arithmetic, in counterpart to non-Euclidean geometries. It is perfectly appropriate to develop formal theories that correspond to the folk wisdom that "1 + 1 = 3 for some combinations of large values of 1 and small values of 3".)
- Set Theory: a fuzzy or relativistic infinity, more arbitrary than Cantor's seemed to be, that can be approached and exceeded gradually, allowing the Archimedean property to be extended to these newly conceived "transfinites" and "transfinitesimals"
- Real Number Theory, etc.: non-vanishing remainders not only give us many prime number based, potentially incommensurate "quantinua", but also naturally derivable infinitesimals or "transfinitesimals", also potentially incommensurate and of arbitrary "order"
- Analysis and "Quantum Micro/Nano/Microcosmic-Topologies": since the "continuum" is replaced by various "quantinua", a "continuous function" might become a "quantinuous function" that must have a slope ≤ 1 if we want the range of the function to take every value in the codomain, and must never skip a value as it increases or decreases. Just as in physics, the "quantinuous" worlds of the very small have their own distinct characteristics, their own "quantum laws of physics" that gradually, "fuzzily" give way to the "classical" worlds of ordinary sizes with their own "classical laws of physics"; and these give way yet again to macrocosmic levels as we fuzzily approach the threshold "non-limits" of our arbitrarily chosen "transfinities" and their own "macrocosmic laws physics". Physicists peering into the cosmos have so far overlooked the implication of quantum and classical physics that the macrocosmic levels of the cosmos must have their own "laws of physics" just as classical levels and microcosmic quantum levels do (although cosmologists have recently been resurrecting Einstein's cosmological constant and begun hunting for inter-

galactic scale repulsive/counter-gravitational forces). This also suggests possibilities for formal theories of “Mega/Giga/Macrocosmic-Topologies”.

The import of The Vanishing Remainders Paradoxes, The Good Shepherd’s Paradoxes, The Bijection Permutation Paradoxes, the Quantinuum Hypothesis, the standard inconsistency of any standard Cantorian set theory, and the desirability of a new and non-standardly non-Cantorian set theory (one in which the Continuum Hypothesis is provably false, as opposed to merely being axiomatically rejected) would seem to be in little doubt.

It was passed over earlier in the mention of what will help, but it will eventually help immensely to explore the historical-psychological development and evolution of Cantorian set theory, along with the rest of mathematics. We need to know “just *how* did it come about?! Just how did we, so many of us and so dedicated to mathematical truth, overlook failings so obvious and so obviously fatal?!” Sufis refer to how unwise men – at times – repent of their sins, but they also refer to how – and hold out the hope to us that – the wise will eventually repent of their heedlessness. Perhaps like Brünnhilde at the end of Götterdämmerung we will become wise.

As noted in the Introduction to this paper, Joseph Warren Dauben, in his intellectual biography of Cantor (*Georg Cantor, His Mathematics and Philosophy of the Infinite*, 1979, Princeton University Press) gives us a fascinating look at the intellectual output of Georg Cantor, a mystic who wanted to find – or create – God, the ultimate paradox, in his mathematics of infinities. He lets us see in great detail that Cantor sought, not to describe, but to approach God through his theory of sets of successively greater “transfinities”, perforce inherently “paradoxical”. Since for Cantor God was/is greater than any infinity, Cantor’s set theory (the very word “theory” derives from the scriptural Greek word for “God”: “Theus”) had to be correspondingly “transcendently infinite”, with transfinities of yet greater transfinities, and an absolute infinity-transfinity, absolutely beyond all those. Since for Cantor God was/is paradox itself, his (or *His*) set theory had to be “transultimately paradoxical”. “The paradise which Cantor created” gives us both, in “goodly”, “Godly” measure.

But there are many other paradises in need of exploration, though we need to construct them first. We are experiencing a Biblical “Destruction of the Temple”. If we want to help “Raise It Up In Three Days”, we will have our work cut out for us. Mathematicians will be kept busy in this new millennium since the need for new and quite non-standardly non-Cantorian set theories and their corresponding number, measure, topological, function and other theories will not... “Vanish”.

> “I do not say that John or Jonathan will realize all this; but such is the character of that morrow which mere lapse of time can never make to dawn. The light which puts out our eyes is darkness to us. Only that day dawns to which we are awake. There is more day to dawn. The sun is but a morning-star.”
> – Henry David Thoreau, *Walden*